Bilateral Gamma Processes in Finance

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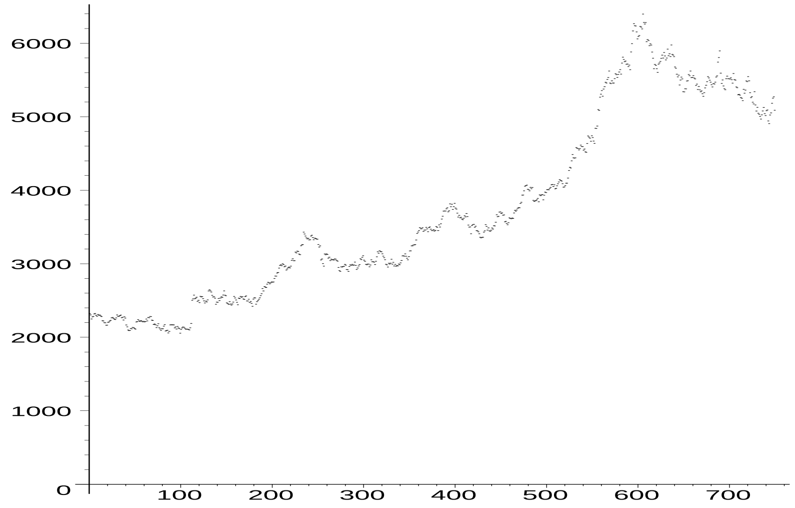
PRisMa 2008
One-Day Workshop on Portfolio Risk Management

September 29th, 2008, Vienna

Introduction



- Stochastic models for risky assets in financial markets.

Exponential Lévy models

- Two financial assets:

$$\begin{cases} S_t = S_0 e^{X_t}, \\ B_t = e^{rt}. \end{cases}$$

- X is a Lévy process.
- Example: Black-Scholes model with $X_t = \sigma W_t + (\mu - \frac{\sigma^2}{2})t$.
- *Idea:* Take $X = X^+ - X^-$, where X^+, X^- are independent subordinators.

Bilateral Gamma distributions

- Four parameters:
  - Shape parameters: $\alpha^+, \alpha^- > 0$,
  - Scale parameters: $\lambda^+, \lambda^- > 0$.

- $\Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ is the distribution of $X^+ - X^-$, where
  - X^+ and X^- are independent,
  - $X^+ \sim \Gamma(\alpha^+, \lambda^+)$,
  - $X^- \sim \Gamma(\alpha^-, \lambda^-)$.

- Family of *bilateral Gamma distributions*.

Bilateral Gamma processes

- Characteristic function:

$$\phi(z) = \left(\frac{\lambda^+}{\lambda^+ - iz} \right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + iz} \right)^{\alpha^-}, \quad z \in \mathbb{R}.$$

- Infinitely divisible with Lévy measure

$$F(dx) = \left(\frac{\alpha^+}{x} e^{-\lambda^+ x} \mathbb{1}_{(0,\infty)}(x) + \frac{\alpha^-}{|x|} e^{-\lambda^- |x|} \mathbb{1}_{(-\infty,0)}(x) \right) dx.$$

- We call the associated Lévy process a *bilateral Gamma process*.

Outline of the talk

- *Bilateral Gamma:*

  1. Bilateral Gamma distributions.
  2. Bilateral Gamma processes.

- *Applications to Finance:*

  1. Option pricing in bilateral Gamma stock models.
  2. Illustration of the theory: DAX 1996-1998.

Related distributions

- Generalized tempered stable distributions (Cont and Tankov 2004):

$$F(dx) = \left(\frac{\alpha^+}{x^{1+\beta^+}} e^{-\lambda^+ x} \mathbb{1}_{(0,\infty)}(x) + \frac{\alpha^-}{|x|^{1+\beta^-}} e^{-\lambda^- |x|} \mathbb{1}_{(-\infty,0)}(x) \right) dx.$$

- CGMY distributions (Carr, Geman, Madan and Yor 1999):

$$F(dx) = \left(\frac{C}{x^{1+Y}} e^{-Mx} \mathbb{1}_{(0,\infty)}(x) + \frac{C}{|x|^{1+Y}} e^{-G|x|} \mathbb{1}_{(-\infty,0)}(x) \right) dx.$$

- Variance Gamma distributions (Madan, Carr and Chang 1998).

Characterization of Variance Gamma distributions

- *Variance Gamma distribution* $VG(\mu, \sigma^2, \nu)$:

$$\phi(z) = \left(1 - iz\mu\nu + \frac{\sigma^2\nu}{2} z^2 \right)^{\frac{1}{\nu}}, \quad z \in \mathbb{R}.$$

- Let $\mu := \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$. There is equivalence between:

1. μ is Variance Gamma;
2. $\alpha^+ = \alpha^-$;
3. X is a time-changed Brownian motion $X_t = W_{Y_t}$;
4. μ is a limit of GH-distributions.

Representation of the density

- Density function for $x > 0$:

$$f(x) = \frac{(\lambda^+)^{\alpha^+}(\lambda^-)^{\alpha^-}}{(\lambda^+ + \lambda^-)^{\frac{1}{2}(\alpha^+ + \alpha^-)}\Gamma(\alpha^+)} \cdot x^{\frac{1}{2}(\alpha^+ + \alpha^-)-1} \cdot e^{-\frac{x}{2}(\lambda^+ - \lambda^-)} \cdot W_{\frac{1}{2}(\alpha^+ - \alpha^-), \frac{1}{2}(\alpha^+ + \alpha^- - 1)}(x(\lambda^+ + \lambda^-)).$$

- $W_{\lambda,\mu}$ denotes the *Whittaker function*

$$W_{\lambda,\mu}(z) = \frac{z^\lambda e^{-\frac{z}{2}}}{\Gamma(\mu - \lambda + \frac{1}{2})} \int_0^\infty t^{\mu-\lambda-\frac{1}{2}} e^{-t} \left(1 + \frac{t}{z}\right)^{\mu+\lambda-\frac{1}{2}} dt, \quad z > 0$$

for $\mu - \lambda > -\frac{1}{2}$.

Properties of the density

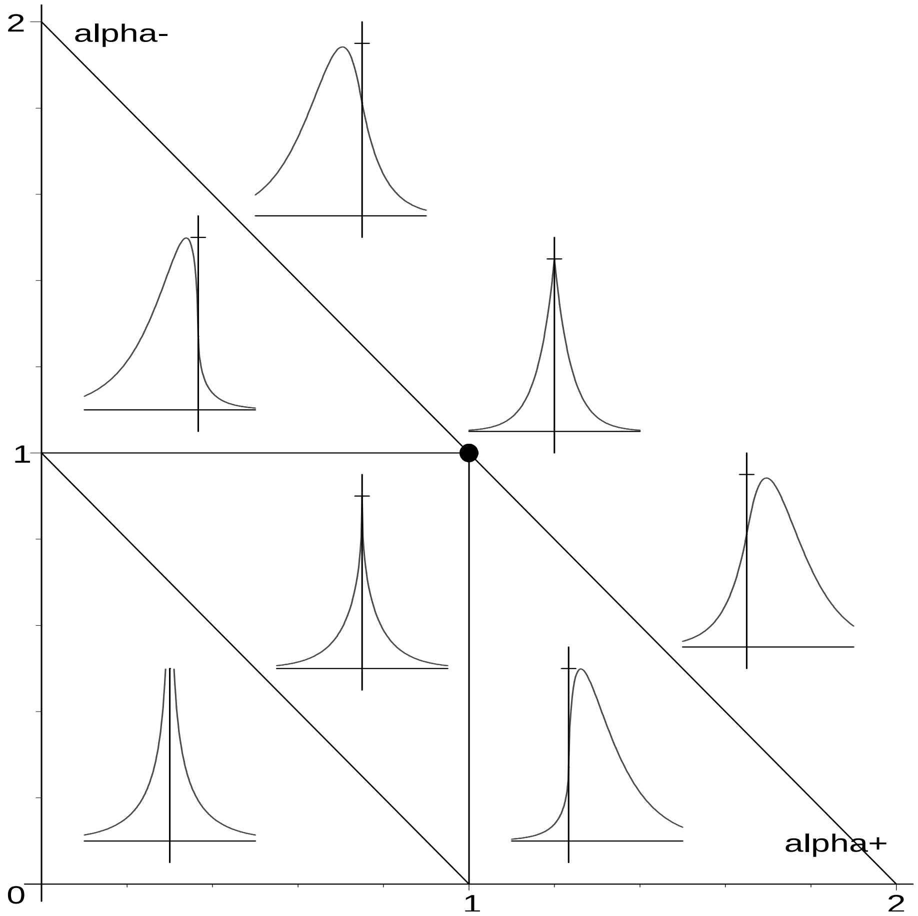
- *Unimodality:* f is strictly increasing/decreasing on $(-\infty, x_0)/(x_0, \infty)$.
- *Smoothness:* Let $N \in \mathbb{N}$ be such that $N < \alpha^+ + \alpha^- \leq N + 1$. Then

$$f \in C^{N-1}(\mathbb{R}) \setminus C^N(\mathbb{R}).$$

- *Semi-heavy tails:* We have the asymptotic behaviour

$$f(x) \sim \begin{cases} C_1 x^{\alpha^+ - 1} e^{-\lambda^+ x} & \text{for } x \to \infty \\ C_2 |x|^{\alpha^- - 1} e^{-\lambda^- |x|} & \text{for } x \to -\infty. \end{cases}$$

Density shapes



Bilateral Gamma processes

- $X = X^+ - X^-$, in particular FV process and no Gaussian part.
- Infinitely many jumps in each compact interval.
- For $0 \leq s < t$ we have:

$$X_t - X_s \sim \Gamma(\alpha^+(t-s), \lambda^+; \alpha^-(t-s), \lambda^-).$$

- Efficient methods for simulating bilateral Gamma processes.

Quick Review

- *Bilateral Gamma distributions:*
  - Simple characteristic function.
  - Densities: unimodal, semi-heavy tailed.
- *Bilateral Gamma processes:*
  - FV sample paths with infinitely many jumps on every interval.
  - Easy to simulate.

Exponential bilateral Gamma models

- Two financial assets:

$$\begin{cases} S_t & = & S_0 e^{X_t}, \\ B_t & = & e^{rt}. \end{cases}$$

- $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$ is a bilateral Gamma process.

- The market is:
  - free of arbitrage,
  - but not complete.

Option Pricing

- Price of a European Call Option:

$$\Pi = \mathbb{E}_{\mathbb{Q}}[(S_T - K)^+],$$

where $\mathbb{Q} \sim \mathbb{P}$ is a martingale measure.

- Fourier transformation: Under "appropriate conditions"

$$\Pi = -\frac{e^{-rT}K}{2\pi} \int_{i\nu-\infty}^{i\nu+\infty} \left(\frac{K}{S_0} \right)^{iz} \frac{\phi_T(-z)}{z(z-i)} dz,$$

where ϕ_T is the characteristic function of X_T under $\mathbb{Q}$.

Requirements on the martingale measure

- There are *several* martingale measures $\mathbb{Q} \sim \mathbb{P}$.
- Under $\mathbb{Q}$, the characteristic function ϕ_T should be "simple".
- Recall that for a bilateral Gamma process:

$$\phi(z) = \left(\frac{\lambda^+}{\lambda^+ - iz} \right)^{\alpha^+} \left(\frac{\lambda^-}{\lambda^- + iz} \right)^{\alpha^-}, \quad z \in \mathbb{R}.$$

- $\mathbb{Q}$ should have an economic interpretation.

Esscher transforms

- For $\theta \in (-\lambda^-, \lambda^+)$ we define $\mathbb{P}^\theta \sim \mathbb{P}$ as

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}}\Big|_{\mathcal{F}_t} := e^{\theta X_t - \Psi(\theta) t}, \quad t \geq 0.$$

- The cumulant generating function Ψ of X is given by

$$\Psi(z) = \alpha^+ \ln\left(\frac{\lambda^+}{\lambda^+ - z}\right) + \alpha^- \ln\left(\frac{\lambda^-}{\lambda^- + z}\right), \quad z \in (-\lambda^-, \lambda^+).$$

Esscher martingale measure

- There exists $\theta \in (-\lambda^-, \lambda^+)$ such that $\mathbb{P}^\theta$ is a martingale measure iff

$$\lambda^+ + \lambda^- > 1.$$

- In this case, it is the unique solution of the equation

$$\left(\frac{\lambda^+ - \theta}{\lambda^+ - \theta - 1}\right)^{\alpha^+} \left(\frac{\lambda^- + \theta}{\lambda^- + \theta + 1}\right)^{\alpha^-} = e^{r-q}, \quad \theta \in (-\lambda^-, \lambda^+ - 1).$$

- We have $X \sim \Gamma(\alpha^+, \lambda^+ - \theta; \alpha^-, \lambda^- + \theta)$.

Relative entropy

- For each $\mathbb{Q} \sim \mathbb{P}$ define the relative entropy

$$\mathbb{H}_{\mathcal{F}_t}(\mathbb{Q} \,|\, \mathbb{P}) = \mathbb{E}_{\mathbb{Q}}\left[\ln \frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t} \right], \quad t \geq 0.$$

- Find a martingale measure $\mathbb{Q}^* \sim \mathbb{P}$ such that

$$\mathbb{H}_{\mathcal{F}_t}(\mathbb{Q}^* \,|\, \mathbb{P}) = \min_{\mathbb{Q} \in \mathrm{EMM}} \mathbb{H}_{\mathcal{F}_t}(\mathbb{Q} \,|\, \mathbb{P}), \quad t \geq 0.$$

- *Minimal entropy martingale measure* (MEMM).

Exponential transform

- Let $\tilde{X}_t := \mathcal{L}(e^{X_t-(r-q)t})$ be the *exponential transform*.
- $\tilde{X}$ is again a Lévy process.
- For $\theta \leq 0$ we define $\mathbb{P}_\theta \sim \mathbb{P}$ as

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}}\Big|_{\mathcal{F}_t} := e^{\theta \tilde{X}_t - \tilde{\Psi}(\theta)t}, \quad t \geq 0.$$

- $\tilde{\Psi}$ denotes the cumulant generating function of $\tilde{X}$.

Minimal entropy martingale measure

- If $\lambda^+ > 1$, there exists $\theta \leq 0$ such that $\mathbb{P}_\theta$ is a MEMM iff

$$\alpha^+ \ln\left(\frac{\lambda^+}{\lambda^+ - 1}\right) + \alpha^- \ln\left(\frac{\lambda^-}{\lambda^- + 1}\right) \geq r - q.$$

- In this case, it is the unique solution of the equation

$$\alpha^+ \int_0^\infty \frac{e^{-\lambda^+ x}}{x}(e^x - 1)e^{\theta(e^x - 1)}dx + \alpha^+ \int_0^\infty \frac{e^{-\lambda^- x}}{x}(e^{-x} - 1)e^{\theta(e^{-x} - 1)}dx$$
$$= r - q, \quad \theta \leq 0.$$

Characteristic function

- X is a Lévy process under $\mathbb{P}_\theta$ with

$$\phi(z) = \exp\bigg(\int_{\mathbb{R}} (e^{izx} - 1) e^{\theta(e^x - 1)} F(dx) \bigg), \quad z \in \mathbb{R}.$$

- The *value* of the minimal relative entropy is given by

$$\begin{aligned} \mathbb{H}_{\mathcal{F}_1}(\mathbb{P}_\theta \,|\, \mathbb{P}) &= r - q \\ &- \alpha^+ \int_0^\infty \frac{e^{-\lambda^+ x}}{x} (e^{\theta(e^x - 1)} - 1) dx - \alpha^- \int_0^\infty \frac{e^{-\lambda^+ x}}{x} (e^{\theta(e^{-x} - 1)} - 1) dx. \end{aligned}$$

Martingale measures considered so far

- *Esscher martingale measure:*
  - Pro: Easy to obtain, X remains bilateral Gamma under $\mathbb{P}^\theta$.
  - Contra: No economic interpretation.
- *Minimal entropy martingale measure:*
  - Pro: Easy to obtain, economic interpretation.
  - Contra: Characteristic function of X under $\mathbb{P}_\theta$ not in closed form.

Bilateral Esscher transforms

- Recall that $X = X^+ - X^-$.
- For $\theta^+ \in (-\infty, \lambda^+)$ and $\theta^- \in (-\infty, \lambda^-)$ we define $\mathbb{P}^{(\theta^+,\theta^-)} \sim \mathbb{P}$ as

$$\frac{d\mathbb{P}^{(\theta^+,\theta^-)}}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} := e^{\theta^+ X_t^+ - \Psi^+(\theta^+)t} \cdot e^{\theta^- X_t^- - \Psi^-(\theta^-)t}, \quad t \geq 0.$$

- Ψ^+, Ψ^- denote the cumulant generating functions of X^+, X^-.

Bilateral Esscher martingale measures

- Define $\Phi : (-\infty, \lambda^+ - 1) \to (-\infty, \lambda^-)$ as

$$\Phi(\theta) := \lambda^- - \left(\left(\frac{\lambda^+ - \theta}{\lambda^+ - \theta - 1} \right)^{\frac{\alpha^+}{\alpha^-}} e^{-(r-q)} - 1 \right)^{-1}.$$

- Then $\mathbb{P}^{(\theta, \Phi(\theta))}$ is a martingale measure for each $\theta \in (-\infty, \lambda^+ - 1)$.
- We have $X \sim \Gamma(\alpha^+, \lambda^+ - \theta; \alpha^-, \lambda^- - \Phi(\theta))$ under $\mathbb{P}^{(\theta, \Phi(\theta))}$. Thus:

$$\Pi = -\frac{e^{-rT} K}{2\pi} \int_{i\nu - \infty}^{i\nu + \infty} \left(\frac{K}{S_0} \right)^{iz} \left(\frac{\lambda^+ - \theta}{\lambda^+ - \theta + iz} \right)^{\alpha^+ T} \left(\frac{\lambda^- - \Phi(\theta)}{\lambda^- - \Phi(\theta) - iz} \right)^{\alpha^- T} \frac{dz}{z(z-i)}.$$

Minimizing the relative entropy

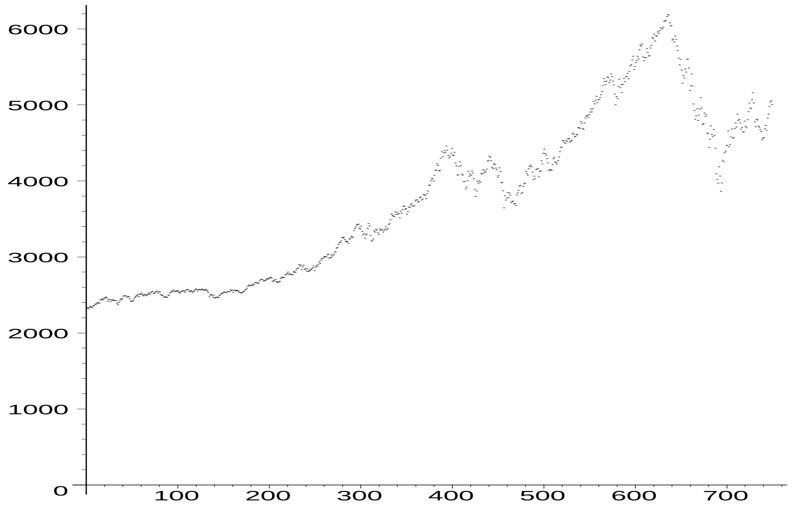
- Find $\theta \in (\infty, \lambda^+ - 1)$ such that

$$\mathbb{H}_{\mathcal{F}_1}(\mathbb{P}^{(\theta,\Phi(\theta))} \,|\, \mathbb{P}) = \min_{\vartheta \in (-\infty, \lambda^+ - 1)} \mathbb{H}_{\mathcal{F}_1}(\mathbb{P}^{(\vartheta,\Phi(\vartheta))} \,|\, \mathbb{P}) \geq \min_{\mathbb{Q} \in \mathrm{EMM}} \mathbb{H}_{\mathcal{F}_1}(\mathbb{Q} \,|\, \mathbb{P}).$$

- The relative entropy is given by:

$$\begin{aligned}\mathbb{H}_{\mathcal{F}_1}(\mathbb{P}^{(\theta,\Phi(\theta))} \,|\, \mathbb{P}) &= \alpha^+ \bigg(\frac{\lambda^+}{\lambda^+ - \theta} - 1 - \ln \bigg(\frac{\lambda^+}{\lambda^+ - \theta} \bigg) \bigg) \\ &+ \alpha^- \bigg(\frac{\lambda^-}{\lambda^- - \Phi(\theta)} - 1 - \ln \bigg(\frac{\lambda^-}{\lambda^- - \Phi(\theta)} \bigg) \bigg), \quad \theta \in (\infty, \lambda^+ - 1).\end{aligned}$$

Example: DAX 1996-1998



- We assume $S_t = S_0 e^{X_t}$, where $X \sim \Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-)$.

Estimates

- Maximum Likelihood Estimate:

$$(\alpha^+, \lambda^+; \alpha^-, \lambda^-) = (1.55, 133.96; 0.94, 88.92).$$

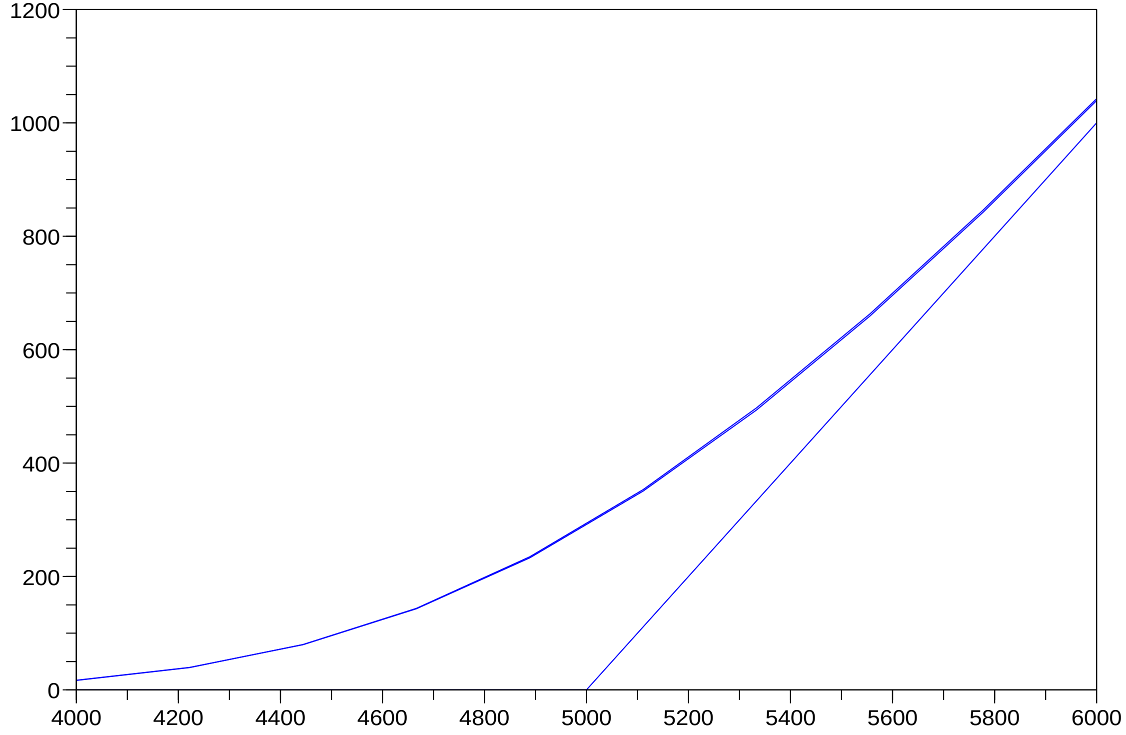
- With $\theta = -5.30$ we have

$$\mathbb{H}_{\mathcal{F}_1}(\mathbb{P}^{(\theta,\Phi(\theta))} \,|\, \mathbb{P}) = \min_{\vartheta \in (-\infty, \lambda^+ - 1)} \mathbb{H}_{\mathcal{F}_1}(\mathbb{P}^{(\vartheta,\Phi(\vartheta))} \,|\, \mathbb{P}) = 0.0029411$$

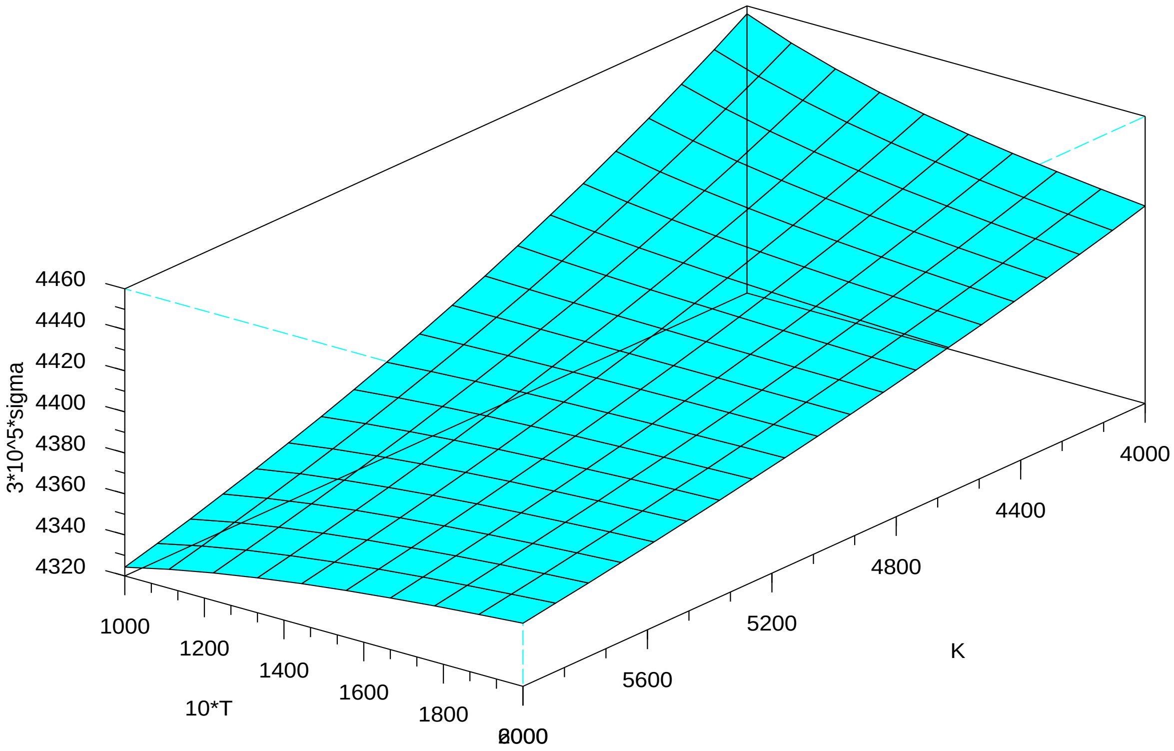
  and $X \sim \Gamma(1.55, 139.26; 0.94, 83.65)$ under $\mathbb{P}^{(\theta,\Phi(\theta))}$.

- Note that $\min_{\mathbb{Q} \in \mathrm{EMM}} \mathbb{H}_{\mathcal{F}_1}(\mathbb{Q} \,|\, \mathbb{P}) = 0.0029409$.

Black Scholes and Bilateral Gamma Prices



Implied volatility surface



Relation to the Normal distribution

- The Central Limit Theorem yields:

$$\Gamma(\alpha^+, \lambda^+; \alpha^-, \lambda^-) \approx N\left(\frac{\alpha^+}{\lambda^+} - \frac{\alpha^-}{\lambda^-}, \frac{\alpha^+}{(\lambda^+)^2} + \frac{\alpha^-}{(\lambda^-)^2}\right).$$

- Berry-Esseen gives us:

$$\sup_{x \in \mathbb{R}} |F_n(x) - \Phi_n(x)| \leq \frac{c}{\sqrt{n}},$$

where $\alpha^+ \rightarrow n\alpha^+$ and $\alpha^- \rightarrow n\alpha^-$ for some $n \in \mathbb{N}$.

Conclusion

- Stock models based on bilateral Gamma processes.
- Option pricing using historical data.
- *Current Research:* Model calibration to option price data.