Modeling and Estimation of Dependent Credit Rating Transitions

Verena Goldammer

Financial and Actuarial Mathematics
Christian Doppler Laboratory for Portfolio Risk Management

Vienna University of Technology

PRisMa 2008 29.09.2008

Outline

Introduction

- Credit ratings describe the credit-worthiness of firms.
- We observe dependent changes of credit ratings of different firms.
- Without modeling the dependence between defaults we underestimate the risk.
- To model dependence, we apply interacting particle systems.

Advantage: Intuitive way to describe the dependence

Applications of Interacting Particle Systems

Model assumption:

Credit ratings follow a time-homogeneous Markov jump process with the dynamics of an interacting particle system.

Applications in the literature:

- Giesecke and Weber (2004): Application of a voter model
- Bielecki and Vidozzi (2006), Frey and Backhaus (2007): Intensity of a credit rating transition for each firm, which depends on the configuration of the credit ratings
- Dai Pra, Runggaldier, Sartori and Tolotti (2007): Mean-field interaction model

Strongly Coupled Random Walk Process

Dynamics:

Independent Poisson processes with intensity $\lambda(x) \geq 0$ for each rating class x.

When Poisson process for rating x jumps, then:

- Rating class y is chosen with probability $P(x, y)$.
- Every firm with rating x tosses a coin with probability p_x of heads, independently of the other firms.
- If head occurs, then the firm changes the rating class from x to y.

Outline

Notation and State Spaces

Notation:

- Credit rating classes: $S = \{1, \ldots, K\}$, where K means the firm is in default and 1 is the best rating class
- Firms: $F = \{1, \ldots, n\}$

Possible state spaces:

1. Assigning a rating to each individual firm:
   - State space S^n

If the firms are indistinguishable, we use the state space:

2. Counting the number of firms in the rating classes:
   - State space: $\mathcal{S} = \left\{\eta \in \{0, \ldots, n\}^S : \sum_{x \in S} \eta(x) = n\right\}$
   - $\eta \in \mathcal{S}$: $\eta(x)$ is the number of firms in rating class x

Parameters of the Credit Rating Process

Parameters:

- $\mu = (\mu_{xy})_{x,y \in S}$: Matrix of transition intensities (Q-matrix) of an individual firm
- $p = (p_x)_{x \in S} \in [0,1]^S$: Dependence vector

If $p_x \in (0,1]$, define:

- The jump intensity of the Poisson process:

$$\lambda(x) = \frac{\mu_x}{p_x}$$

  $\mu_x = -\mu_{xx}$: Intensity of a single firm to leave rating x
- The probability of a rating change from x to y:

$$P(x,y) = \frac{\mu_{xy}}{\mu_x}, \quad x \neq y,\ \mu_x > 0$$

Process with State Space S^n

Process with state space S^n:

- $(X_t)_{t \geq 0}$: Markov jump process with state space S^n, describing the individual credit ratings of the n firms.
- X has the dynamics of the strongly coupled random walk.

Transition intensities:

- Independent Poisson processes at the rating classes
  $\Rightarrow$ Intensity of a change of $k \geq 2$ firms
  with different ratings is zero.
- Intensity of a change of $k \geq 2$ firms to different ratings is zero.

Definition of Feasible Transitions of $(X_t)_{t \geq 0}$

- **Feasible transition:**
  The credit rating of B firms changes from one rating class x to another rating class $y \neq x$, where A firms had originally rating x.
- Intensity of such a feasible transition:

$$\lambda(x) P(x, y) p_x^B (1 - p_x)^{A-B}$$

- Using the matrix μ of transition intensities of a single firm:

$$\mu_{xy} p_x^{B-1} (1 - p_x)^{A-B}$$

If $p_x = 0$, then the intensity of a change of exactly one firm is μ_{xy} and zero otherwise.
$\Rightarrow$ The firms with rating x move independently.

Q-Matrix of the Process with State Space S^n

Notation:

- $z, \tilde{z} \in S^n$: Rating configurations
- Transition $z \to \tilde{z}$ feasible:
  B firms change from rating class x to rating class $y \neq x$, where A firms had originally rating x.
- A_u: Number of firms with rating u in z

Q-Matrix:

$$Q_n(z, \tilde{z}) = \begin{cases} \mu_{xy} p_x^{B-1} (1 - p_x)^{A-B}, & \text{if } z \to \tilde{z} \text{ is feasible,} \\ -\sum_{u \in S} \mu_u \sum_{j=0}^{A_u - 1} (1 - p_x)^j, & \text{if } z = \tilde{z}, \\ 0 & \text{otherwise.} \end{cases}$$

DSP

Process with State Space $\mathcal{S}$ (Indist. Firms)

- $(\eta_t)_{t\geq 0}$: Markov jump process with state space $\mathcal{S}$, which describes the number of firms in the rating classes
- η_{xy}^k: In configuration η, k firms with rating x change to $y \neq x$.
- Intensity of a change from η to η_{xy}^k:

$$Q^{\mathcal{S}}(\eta, \eta_{xy}^k) = \binom{\eta(x)}{k} \mu_{xy} p_x^{k-1} (1 - p_x)^{\eta(x)-k}$$

- Q-matrix of the process:

$$Q^{\mathcal{S}}(\eta, \eta') = \begin{cases} Q^{\mathcal{S}}(\eta, \eta_{xy}^k), & \text{if } \eta' = \eta_{xy}^k \text{ for } x, y \in S \\ & \text{and } k \in \{1, \ldots, \eta(x)\}, \\ -\sum\limits_{\substack{x,y \in S \\ x \neq y}} \sum\limits_{k=1}^{\eta(x)} Q^{\mathcal{S}}(\eta, \eta_{xy}^k), & \text{if } \eta = \eta', \\ 0, & \text{otherwise.} \end{cases}$$

Preserving the Markov Property

Lemma (Function of MP again MP)

Assumptions:

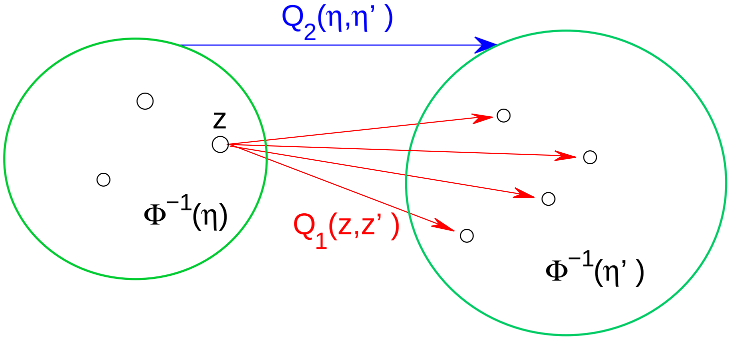
- S_1, S_2: *Finite state spaces*
- $\Phi : S_1 \to S_2$: *Arbitrary function*
- $(X_t)_{t \geq 0}$: *Markov jump process w.r.t. the filtration* $(\mathcal{F}_t)_{t \geq 0}$, *generated by a Q-matrix* Q_1 *and state space* S_1
- Q_2: *Q-matrix, such that for all* $\eta, \eta' \in S_2$ *with* $\eta \neq \eta'$

$$Q_2(\eta, \eta') = \sum_{z' \in \Phi^{-1}(\eta')} Q_1(z, z'), \quad \textit{for all } z \in \Phi^{-1}(\eta)$$

$\Rightarrow$ $(\Phi(X_t))_{t \geq 0}$ *is a Markov jump process w.r.t.* $(\mathcal{F}_t)_{t \geq 0}$ *with state space* S_2, *generated by the Q-matrix* Q_2.

Illustration of the Condition

$$Q_2(\eta, \eta') = \sum_{z' \in \Phi^{-1}(\eta')} Q_1(z, z'), \quad \text{for all } z \in \Phi^{-1}(\eta)$$



Strongly Coupled Random Walk is Dynamics of $(X_t)_{t\geq 0}$

Theorem

Assumptions:

- $(X_t)_{t\geq 0}$: *Markov jump process with Q-matrix* Q_n *and state space* S^n
- $\Phi : S^n \to \mathcal{S}$, *where for* $z = (z_1, \ldots, z_n) \in S^n$

$$(\Phi z)(x) = \sum_{i=1}^{n} \mathbb{1}_{\{z_i = x\}}, \quad x \in S$$

$\Rightarrow$ $\eta_t = \Phi(X_t)$ *for* $t \geq 0$ *is a Markov jump process with state space* $\mathcal{S}$ *and Q-matrix* $Q^{\mathcal{S}}$.

Embedding of a Model with Fewer Firms

Theorem

Assumptions:

- $m, n \in \mathbb{N}$ *with* $m < n$*: Number of firms*
- $(X_t)_{t \geq 0}$*: Markov jump process with state space* S^n *and Q-matrix* Q_n
- $\pi : S^n \rightarrow S^m$*: Projection with* $\pi(x) = x|_{S^m}$

$\Rightarrow$ $Y_t = \pi(X_t)$ *for* $t \geq 0$ *is a Markov jump process with state space* S^m*, generated by Q-matrix* Q_m*.*

Intensity of a rating change of m firms in a model with n firms
$=$ Intensity of a change in a model with m firms

Outline

Portfolio and Profit–and–Loss Function

Portfolio:

- n defaultable zero-coupon bonds issued by n firms with face value f_i and maturity T_i for $i = 1, \ldots, n$
- Simplification: No recovery and zero default-free interest rate

Portfolio value at future time $t \leq \min\{T_1, \ldots, T_n\}$:

$$V(t) = \sum_{i=1}^{n} f_i \, \mathbf{P}[X_i(T_i) \neq K | X_t] = \sum_{i=1}^{n} f_i \left(1 - \left(\exp\{\mu(T_i - t)\}\right)_{X_i(t),K}\right)$$

Profit–and–loss during $(0, t]$:

$$\text{P\&L}(t) = V(t) - V(0)$$

Simulation of the Loss of the Portfolio

Model assumptions:

- $K = 8$ credit rating classes
- Initial rating: All firms have rating 4.
- Movement according to our model with generator μ based on data of Standard & Poor's
- Dependence vector $(p_x)_{x \in S}$: $p_x = p$ for all $x \in S$

Portfolio assumptions:

- Number of bonds: $n = 100$
- Face value: $f_i = 1$ for every bond
- Common maturity: $T_i = 10$ years

Distribution Function of Profit and Loss

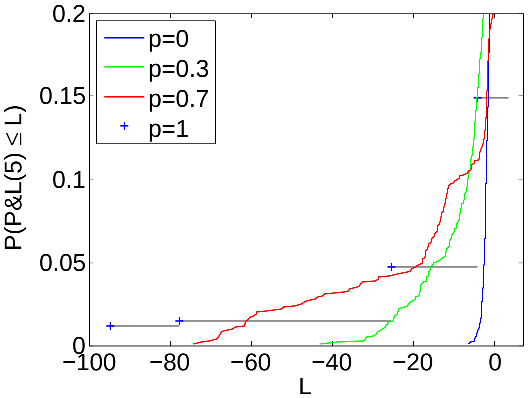


Figure: Distribution of the profit and loss of the portfolio at time $t = 5$ for $n = 100$ firms (1 000 simulations). The dependence parameter p_x equals p for each rating class. Initial value of the portfolio is $V(0) = 95$.

Distribution Function of Profit and Loss

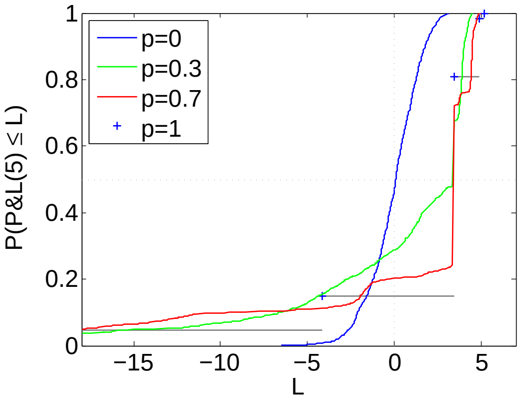


Figure: Distribution of the profit and loss of the portfolio at time $t = 5$ for $n = 100$ firms (1 000 simulations). The dependence parameter p_x equals p for each rating class. Initial value of the portfolio is $V(0) = 95$.

Outline

Rating Process with Varying Number of Firms

Notation:

- $(X_{t,n})_{t\geq 0}$ for $n \in \mathbb{N}$: Markov jump process with state space S^n and Q-matrix Q_n
- $X_{0,n}$: Initial rating, independent of the parameters p and μ
- N: Random number of firms, independent of $(X_{t,n})_{t\geq 0}$ for all $n \in \mathbb{N}$ and independent of the parameters p and μ

Credit rating process with N firms:

$$(Y_t)_{t\geq 0} = (X_{t,N})_{t\geq 0}$$

We observe independent samples of the credit rating process $(Y_t)_{t\in[0,T)}$.

Space of the Sample Paths of $X_{t,n}$

Take modification of Y with piecewise constant sample paths with finite number of jumps

$\Rightarrow$ Sample path of the process $(X_{t,n})_{t\in[0,T)}$, which jumps m times in $[0,T)$:

$$((z_0, t_0), \ldots, (z_{m-1}, t_{m-1}), z_m)$$

That means:

- Process starts in z_0 and remains there for time t_0.
- Then it jumps to z_1 and remains there for time t_1.

  $\vdots$

- Finally it reaches z_m and stays there until T.

Space of the Sample Paths of Y

Space of the sample paths of $(X_{t,n})_{t\in[0,T)}$ with m jumps:

$$\mathcal{S}_{n,m} = (S^n \times \mathbb{R})^m \times S^n \quad \text{for } m \in \mathbb{N}$$
$$\mathcal{S}_{n,0} = S^n \quad \text{for } m = 0$$

Space of sample paths of $(X_{t,n})_{t\in[0,T)}$:

$$\mathcal{S}_n = \bigcup_{m=0}^{\infty} \mathcal{S}_{n,m}$$

Space of sample paths of the process $Y = X_N$:

$$\mathcal{S}^Y = \bigcup_{n=1}^{\infty} \mathcal{S}_n$$

Measure on Space of Sample Paths

- $\mathcal{V}$: Lebesgue–Borel measure on $\mathbb{R}$
- $\mathcal{N}_n$: Counting measure on S^n
- $\sigma_{n,m}$: Product measure on $\mathcal{S}_{n,m}$ with

$$\sigma_{n,m} = (\mathcal{N}_n \otimes \mathcal{V})^{\otimes m} \otimes \mathcal{N}_n$$

- $\mathcal{B}$: Smallest σ-algebra containing all sets of $\mathcal{S}^Y$ whose intersection with $\mathcal{S}_{n,l}$ is a Borel set for each pair $(n, l) \in \mathbb{N} \times \mathbb{N}_0$

Define the measures:

$$\sigma_n(C \cap \mathcal{S}_n) = \sum_{m=0}^{\infty} \sigma_{n,m}(C \cap \mathcal{S}_{n,m}) \quad \text{for } C \in \mathcal{B}$$

and

$$\sigma(C) = \sum_{n=1}^{\infty} \sigma_n(C \cap \mathcal{S}_n) \quad \text{for } C \in \mathcal{B}$$

Feasible Path of $X_{t,n}$

Feasible path of $(X_{t,n})_{t\in[0,T)}$:

- Start in z_0 is possible, i.e. $\mathbf{P}(X_{0,n} = z_0) > 0$.
- All transitions are feasible.

Define for feasible path ω:

- $N^{(\omega)}_{x,y,A,B}$: Number of simultaneous transitions of B firms from rating class x to $y \neq x$, A firms originally with rating x
- $T^{(\omega)}_{x,A}$: Total time that exactly A firms have rating x

Density of a Sample Path

Theorem

Let E be a measurable subset of $\mathcal{S}^Y$. Then for $n \in \mathbb{N}$ with $\mathbf{P}(N = n) > 0$

$$\mathbf{P}\Big((Y_t)_{t\in[0,T)} \in E \Big| N = n\Big) = \int_{E\cap\mathcal{S}_n} f_{Q_n}(\omega)\,\sigma_n(d\omega),$$

where $f_{Q_n}(\omega) = 0$*, if the path is not feasible,*
and for a feasible path $\omega \in \mathcal{S}_n$

$$f_{Q_n}(\omega) = \mathbf{P}(X_{0,n} = z_0)\exp\left\{ -\sum_{A=1}^{n}\sum_{x\in S} T_{x,A}^{(\omega)}\,\mu_x \sum_{j=0}^{A-1}(1-p_x)^j \right\}$$
$$\times \prod_{\substack{x,y\in S\\ x\neq y}} \prod_{\substack{A,B=1\\ A\geq B}}^{n} \left(\mu_{xy}\, p_x^{B-1}(1-p_x)^{A-B}\right)^{N_{x,y,A,B}^{(\omega)}}.$$

◂ Qn

Likelihood Function

- $\omega_1, \ldots, \omega_k \in \mathcal{S}^Y$: Feasible sample paths
- n_j: Number of firms in path ω_j
- $n := \max_{j \in \{1,\ldots,k\}} n_j$: Maximal number of observed firms
- $N_{x,y,A,B}$: Total number of simultaneous rating changes of B firms from x to $y \neq x$, A firms originally with rating x
- $T_{x,A}$: Total time that exactly A firms have rating x

Conditional likelihood function, given $N_1 = n_1, \ldots, N_k = n_k$:

$$\mathcal{L}(\omega_1, \ldots, \omega_k) = \prod_{\substack{x,y \in S \\ x \neq y}} \prod_{\substack{A,B=1 \\ A \geq B}}^{n} \left(\mu_{xy} p_x^{B-1} (1-p_x)^{A-B} \right)^{N_{x,y,A,B}}$$

$$\times \left(\prod_{j=1}^{k} \mathbf{P}(X_{0,n_j} = z_0^j) \right) \exp\left\{ -\sum_{x \in S} \mu_x \sum_{A=1}^{n} T_{x,A} \sum_{j=0}^{A-1} (1-p_x)^j \right\}$$

Credit Rating Sets

Subsets of the rating set $S = \{1, \ldots, K\}$, depending on the observed feasible paths of Y:

- $x \in S^{T>0}$, iff at least one firm has rating x temporarily during the observation period
  $\Rightarrow$ Parameter estimation is possible
- $x \in S^{p>0}$, iff at least two firms with rating x change the rating simultaneously at some time
  $\Rightarrow$ Firms change the rating x not independently
- $x \in S^{p<1}$, iff at least at one time not all firms with rating x change this rating together
  $\Rightarrow$ Firms are not totally dependent

Maximum Likelihood Estimator for the Q-Matrix μ

Theorem (MLE for μ)

A maximum likelihood estimator (MLE) of the parameter vector $[p, \mu]$ is given as follows.
(a) For $(x, y) \in S^{T>0} \times S$ with $x \neq y$ the MLE of entry μ_{xy}:

$$\hat{\mu}_{xy} = \frac{\sum_{A=1}^{n} \sum_{B=1}^{A} N_{x,y,A,B}}{\sum_{A=1}^{n} T_{x,A} \sum_{j=0}^{A-1} (1 - \hat{p}_x)^j},$$

where $\hat{p}_x$ is the MLE of p_x, which is given in the next slides.

$\Rightarrow$ If $p_x = 0$, then we get the same MLE in this model for μ as in the independent model:

$$\hat{\mu}^{\text{ind}}_{xy} = \frac{\sum_{A=1}^{n} N_{x,y,A,1}}{\sum_{A=1}^{n} A \, T_{x,A}}$$

Max. Likelihood Estimator for Dep. Parameter p_x

Theorem (MLE for p)

(b) For the rating classes $x \in S^{p>0} \cap S^{p<1}$ the MLE of p_x is uniquely determined. The MLE is the unique root in $(0,1)$ of the polynomial of degree $\leq n$ with coefficients

$$c_0 = \sum_{\substack{A,B=1 \\ A \geq B}}^{n} (B-1)\widetilde{N}_{x,A,B} \sum_{i=1}^{n} i \; T_{x,i}$$

$$c_j = (-1)^j \sum_{\substack{A,B=1 \\ A \geq B}}^{n} \widetilde{N}_{x,A,B} \sum_{i=j}^{n} \binom{i}{j} \left(\frac{i-j}{j+1} B + A - i \right) T_{x,i}$$

for $j \in \{1, \ldots, n\}$.

$\widetilde{N}_{x,A,B}$: *Total number of rating changes of B firms with rating x, A firms originally with rating x*

Max. Likelihood Estimator for Dep. Parameter p_x

Theorem (MLE for p)

(c) For $x \in S^{p>0} \setminus S^{p<1}$ the MLE is $\hat{p}_x = 1$.

In the remaining case of insufficient observations:

(d) If $x \in S^{p<1} \setminus S^{p>0}$, the MLE for p_x is one of the following:

- *$\hat{p}_x = 0$ or*
- *the unique root of the polynomial with coefficients as above.*

(e) For $x \in S^{T>0} \setminus (S^{p>0} \cup S^{p<1})$ the MLE is $\hat{p}_x = 1$, if

- *there is a change, where only one firm has rating x and is leaving x, and*
- *at least two firms have rating x for a time interval.*

Further Research

Asymptotic properties of the estimator:

- Consistency?
- Asymptotical normality?

Extension of the model:

- Observations of the credit ratings only at discrete time points

$\Rightarrow$ Estimation of the parameters using historical rating transitions

- Independent copies of the process, each one symbolizing an sector of industry

References

T. Bielecki, A. Vidozzi and L. Vidozzi.
An efficient approach to valuation of credit basket products and ratings triggered step-up bonds,
Working paper, Illinois Institute of Technology, 2006

R. Frey and J. Backhaus.
Pricing and hedging of portfolio credit derivatives with interacting default intensities,
Working paper, University of Leipzig, 2007

K. Giesecke and S. Weber.
Cyclical correlation, credit contagion, and portfolio losses,
Journal of Banking and Finance, 2004

References

P. Dai Pra, W. Runggaldier, E. Sartori and M. Tolotti.
Large portfolio losses; a dynamic contagion model,
Working paper, University of Padova, 2007.

F. Spitzer.
Infinite systems with locally interacting components,
The Annals of Probability, 1981.