Strong Taylor approximation of SDEs and application to the Lévy LIBOR model

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## Derivatives markets

According to the BIS Quarterly Review, the notional amounts outstanding for OTC derivatives in billions of US$, were:

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Foreign exchange</td>
<td>40,239</td>
<td>48,620</td>
<td>56,238</td>
</tr>
<tr>
<td><strong>Interest rate</strong></td>
<td>291,115</td>
<td>346,937</td>
<td>393,138</td>
</tr>
<tr>
<td>Equity-linked</td>
<td>7,488</td>
<td>9,202</td>
<td>8,509</td>
</tr>
<tr>
<td>Commodity</td>
<td>7,115</td>
<td>7,567</td>
<td>9,000</td>
</tr>
<tr>
<td>Credit default swaps</td>
<td>28,650</td>
<td>42,580</td>
<td>57,894</td>
</tr>
<tr>
<td>Unallocated</td>
<td>39,682</td>
<td>61,501</td>
<td>71,225</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>414,290</td>
<td>516,407</td>
<td>596,004</td>
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- Interest rate derivatives are an important part of global financial markets
Interest rates evolution

Evolution of interest rate term structure, 2003 – 2004 (picture: Th. Steiner)
Basic interest rates – Notation

\( B(t, T) \) time-\( t \) price of a zero coupon bond for maturity \( T \), i.e. \( B(T, T) = 1 \);

\( L(t, T) \) time-\( t \) forward LIBOR for \([T, T + \delta] \):

\[
L(t, T) = \frac{1}{\delta} \left( \frac{B(t, T)}{B(t, T + \delta)} - 1 \right)
\]

\( F(t, T, U) \) time-\( t \) forward price for maturities \( T \) and \( U \): \( F(t, T, U) = \frac{B(t, T)}{B(t, U)} \)

Relationships:

\[
F(t, T, T + \delta) = \frac{B(t, T)}{B(t, T + \delta)} = 1 + \delta L(t, T)
\] (1)
Modeling approaches

Short rate models, Forward rate models (HJM), . . .

**LIBOR market models:** LIBOR rates are modeled directly as

$$L(t, T_k) = L(0, T_k) \mathcal{E}\left( \int_0^t \lambda(s, T_k) dW^{T_{k+1}}_s \right)_t$$

for all $T \in \{T_0, T_1, T_2, \ldots, T_{N-1}, T_N\}$, a discrete tenor structure. Here $W^{T_{k+1}}$ is a Brownian motion under the forward martingale measure $\mathbb{P}^{T_{k+1}}$. Forward measures are related via

$$\frac{d\mathbb{P}^{T_{k+1}}}{d\mathbb{P}^{T_k}} = \frac{F(T_k, T_k, T_{k+1})}{F(0, T_k, T_{k+1})}.$$ 

**Advantages:** positive LIBOR rates; consistent with market practice i.e. Black 1976. Calibration? Tractability?
Caplet implied volatilities are constant neither across strike nor across maturity!
Time-inhomogeneous Lévy processes

Let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) be a complete stochastic basis, where \(\mathcal{F} = \mathcal{F}_{T^*}\) and \(\mathbb{F} = (\mathcal{F}_t)_{t \in [0, T^*]}\).

A time-inhomogeneous Lévy process, is an adapted, càdlàg real-valued stochastic process \(H = (H_t)_{0 \leq t \leq T^*}\) with \(H_0 = 0\) a.s., such that:

**D1** \(H\) has independent increments, i.e. \(H_t - H_s\) is independent of \(\mathcal{F}_s, 0 \leq s < t \leq T^*\),

**D2** the law of \(H_t\) is described by the characteristic function

\[
\mathbb{E} \left[ e^{iuH_t} \right] = \exp \left( \int_0^t \kappa_s(iu) \, ds \right),
\]

\[
\kappa_s(iu) = iu b_s - \frac{u^2 c_s}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux) F_s(dx),
\]

for all \(t \in [0, T^*]\). Here \(b_s \in \mathbb{R}, c_s \in \mathbb{R}_{\geq 0}\) and \(F_s\) are Lévy measures, \(s \in [0, T^*]\).
Assumption (A). The triplets \((b_s, c_s, F_s)\) satisfy:

\[
\int_0^{T_*} \left( |b_s| + c_s + \int_{\mathbb{R}} (1 \wedge x^2) F_s(dx) \right) ds < \infty.
\]

\(\blacktriangleright\) \(H\) is a semimartingale on \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\). [FTAP applies]

Assumption (EM). There exist \(M, \varepsilon > 0\), such that the Lévy measures \(F_s\) satisfy:

\[
\int_0^{T_*} \int_{\{|x| > 1\}} e^{ux} F_s(dx) ds < \infty, \quad \forall u \in [-{(1 + \varepsilon)}M, {(1 + \varepsilon)}M] =: \mathbb{M}.
\]

Moreover, \(\int_{\{|x| > 1\}} e^{ux} F_s(dx) < \infty\) for all \(s \in [0, T_*]\) and all \(u \in \mathbb{M}\).

\(\blacktriangleright\) \(\kappa_s\) can be extended to \(\mathbb{R} \times i\mathbb{M}\), for all \(s \in [0, T_*]\). [Option pricing and Greeks]
Semimartingale semantics

The process $H$ is a special semimartingale with canonical decomposition

$$H_t = \int_0^t b_s \, ds + \int_0^t \sqrt{c_s} \, dW_s + \int_0^t \int_\mathbb{R} x(\mu - \nu)(ds, dx)$$  \hspace{1cm} (4)$$

and the triplet of predictable characteristics is $(B, C, \nu)$, where

$$B_t = \int_0^t b_s ds, \quad C_t = \int_0^t c_s ds, \quad \nu([0, t] \times A) = \int_0^t \int_A F_s(dx)ds,$$  \hspace{1cm} (5)$$

$A \in \mathcal{B}(\mathbb{R})$. $W$ is a $\mathcal{F}$-Brownian motion, $\mu$ is the random measure of jumps of $H$ and $\nu$ is the compensator of $\mu$.

$(B, C, \nu)$ completely determines the distribution of $H$. 

$\blacksquare$
Absolute errors in caplet calibration (3-“factor” time-inhomogeneous Lévy process)
The Lévy LIBOR model – construction

Discrete tenor structure: $T_0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T_\ast$, $\delta_i = T_i - T_{i-1}$.

(LR1): For any maturity $T_i$ there exists a bounded, continuous, deterministic volatility function $\lambda(\cdot, T_i) : [0, T_i] \to \mathbb{R}$, corresponding to $L(\cdot, T_i)$. Moreover,$$
\sum_{i=1}^N |\lambda(s, T_i)| \leq M, \quad \forall s \in [0, T_\ast].$$

(LR2): Assume a strictly positive and strictly decreasing initial term structure $B(0, T)$, $T \in (0, T_\ast]$. Consequently

$$L(0, T) = \frac{1}{\delta} \left( \frac{B(0, T)}{B(0, T + \delta)} - 1 \right) > 0.$$
Let $\mathbb{P}_{T_*}$ be the terminal forward martingale measure. Postulate that the LIBOR with the longest maturity has dynamics

$$L(t, T_N) = L(0, T_N) \exp \left( \int_0^t b^L(s, T_N) ds + \int_0^t \lambda(s, T_N) dH_s \right), \quad (6)$$

where $H$ is a $\mathbb{P}_{T_*}$-Lévy process, with canonical decomposition

$$H = \int_0^t \sqrt{c_s} dW^T_s + \int_0^t \int_\mathbb{R} x(\mu^H - \nu^{T_*})(ds, dx) \quad (7)$$

and the drift term is

$$b^L(s, T_N) = -\frac{1}{2} \lambda^2(s, T_N) c_s - \int_\mathbb{R} \left( e^{\lambda(s, T_N)x} - 1 - \lambda(s, T_N)x \right) F_s^{T_*}(dx). \quad (8)$$
Equivalently,

\[ L(t, T_N) = L(0, T_N) \mathcal{E}\left(Z(T_N)\right)_t, \tag{9} \]

where \( Z(T_N) \) is the exponential transform of \( \int_0^t \lambda(s, T_N) dH_s \), i.e.

\[ Z(T_N) = \int_0^t \lambda(s, T_N) dH_s + \int_0^t \int_\mathbb{R} (e^{\lambda(s, T_N)x} - 1 - \lambda(s, T_N)x) \mu^H(ds, dx). \tag{10} \]

Therefore, \( L(\cdot, T_N) \) satisfies the SDE

\[ dL(t, T_N) = L(t-, T_N)dZ_t(T_N), \tag{11} \]

with initial condition \( L(0, T_1) = \frac{1}{\delta} \left( \frac{B(0, T_N)}{B(0, T)} - 1 \right). \)
Define the forward measure \( \mathbb{P}_{T_N} \) via

\[
\frac{d\mathbb{P}_{T_N}}{d\mathbb{P}_{T_*}} = \frac{F(T_N, T_N, T_*)}{F(0, T_N, T_*)} = 1 + \frac{\delta L(T_N, T_N)}{1 + \delta L(0, T_N)}. \tag{12}
\]

Then, the dynamics of the forward rate is

\[
dF(t, T_N, T_*) = \delta dL(t, T_N)
\]

\[
= F(t-, T_N, T) \left( \frac{\delta L(t-, T_N)}{1 + \delta L(t-, T_N)} \lambda(t, T_N) dH_t \right.
\]

\[
+ \left. \frac{\delta L(t-, T_N)}{1 + \delta L(t-, T_N)} \int_{\mathbb{R}} (e^{\lambda(t, T_N)x} - 1 - \lambda(s, T_N)x) \mu^H(dt, dx) \right). \tag{13}
\]
Now, the next LIBOR rate has dynamics

\[ L(t, T_{N-1}) = L(0, T_{N-1}) \exp \left( \int_0^t b^L(s, T_{N-1}) ds + \int_0^t \lambda(s, T_{N-1}) dH^T_N \right), \tag{14} \]

where \( H^T_N \) is a semimartingale with canonical decomposition

\[ H^T_N = \int_0^\cdot \sqrt{c_s} dW^T_N + \int_0^\cdot \int_{\mathbb{R}} x(\mu^H - \nu^T_N)(ds, dx); \tag{15} \]

while \( b^L(\cdot, T_{N-1}) \) ensures martingality. Here the \( \mathbb{IP}_{T_N} \)-Brownian motion is

\[ W^T_N = W^T_* - \int_0^\cdot \frac{\delta L(s-, T_N)}{1 + \delta L(s-, T_N)} \lambda(s, T_N) \sqrt{c_t} ds, \tag{16} \]

and the \( \mathbb{IP}_{T_N} \)-compensator of \( \mu^H \) is

\[ \nu^T_N(dt, dx) = \left( \frac{\delta L(t-, T_N)}{1 + \delta L(t-, T_N)}(e^{\lambda(t, T_N)x} - 1) + 1 \right) \cdot \nu^T_*(dt, dx). \tag{17} \]
The construction proceeds inductively; the dynamics of the LIBOR $L(\cdot, T_i)$ under the forward measure $\mathbb{P}_{T_{i+1}}$, is

$$L(t, T_i) = L(0, T_i) \exp \left( \int_0^t b^L(s, T_i) \, ds + \int_0^t \lambda(s, T_i) \, dH^{T_{i+1}}_s \right), \quad (18)$$

where $b^L(\cdot, T_i)$ ensures the martingality of the rate, and

$$H^{T_{i+1}} = \int_0^\cdot \sqrt{c_s} \, dW^{T_{i+1}}_s + \int_0^\cdot \int_{\mathbb{R}} x(\mu^H - \nu^{T_{i+1}})(ds, dx). \quad (19)$$

$W^{T_{i+1}}$ is a $\mathbb{P}_{T_{i+1}}$-Brownian motion and the $\mathbb{P}_{T_{i+1}}$-compensator of $\mu^H$ is

$$\nu^{T_{i+1}}(ds, dx) = \prod_{l=i+1}^N \frac{\delta L(s-T_l)}{1 + \delta L(s-T_l)} \left( (e^{\lambda(s, T_l)} x - 1) + 1 \right) \nu^T(ds, dx). \quad (20)$$

Consequence: $H^{T_{i+1}}$ is not a time-inhomogeneous Lévy process (unless $i = N$).
Frozen compensator approximation

Idea: replace the random terms by their deterministic initial values

\[ \frac{\delta L(t-, T_l)}{1 + \delta L(t-, T_l)} \approx \frac{\delta L(0, T_l)}{1 + \delta L(0, T_l)} \]

in the compensator \( \nu^{T_{i+1}} \) of \( \mu^H \) under the forward martingale measure \( \mathbb{P}_{T_{i+1}} \); i.e.

\[ \nu^{0, T_{i+1}}(ds, dx) = \prod_{l=i+1}^{N} \frac{\delta L(0, T_l)}{1 + \delta L(0, T_l)} \left( \left( e^{\lambda(s, T_l)x} - 1 \right) + 1 \right) \nu^{T_*}(ds, dx). \tag{21} \]

- \( H^{0, T_{i+1}} \) is a time-inhomogeneous Lévy process.

- Pricing is possible (but hard). Simulation is very hard ...

- What is the error? The error is compounded ...
Terminal measure dynamics

We lift all LIBOR rates to the terminal measure $\mathbb{P}_{T*}$:

$$L(t, T_i) = L(0, T_i) \exp \left( \int_0^t \bar{b}(s, T_i) \, ds + \int_0^t \lambda(s, T_i) \, dH_s \right),$$  \hspace{1cm} (22)

where $H$ is the driving time-inhomogeneous Lévy process, and

$$\bar{b}(s, T_i) = -\frac{1}{2} \lambda^2(s, T_i) c_s - c_s \lambda(s, T_i) \sum_{l=i+1}^{N} \frac{\delta_l L(s-, T_l)}{1 + \delta_l L(s-, T_l)} \lambda(s, T_l)$$

$$- \int_{\mathbb{R}} \left( e^{\lambda(s,T_i)x} - 1 \right) \prod_{l=i+1}^{N} \beta(s, x, T_l) - \lambda(s, T_i)x \right) \, F_{s}^{T*}(dx).$$  \hspace{1cm} (23)

This can also be expressed as a stochastic exponential.
Under the terminal measure, the dynamics of $L(\cdot, T_i)$ satisfy the SDE

$$dL(t, T_i) = L(t-, T_i) dZ_t$$

(24)

where $Z = (Z_t)_{0 \leq t \leq T}$ is a semimartingale with canonical decomposition

$$Z = \int_0^t b(s, T_i) ds + \int_0^t \lambda(s, T_i) dH_s$$

$$+ \int_0^t \int_0^t (e^{\lambda(s,T_i)x} - 1 - \lambda(s,T_i)x) \mu^H(ds, dx)$$

(25)

and the (random) drift term is

$$b(s, T_i) = -c_s \lambda(s, T_i) \sum_{l=i+1}^N \frac{\delta_l L(s-, T_l)}{1 + \delta_l L(s-, T_l)} \lambda(s, T_l)$$

$$- \int_{\mathbb{R}} \left( e^{\lambda(s,T_i)x} - 1 \right) \prod_{l=i+1}^N \beta(s, x, T_l) - \lambda(s, T_i)x \right) F_{s}^{T*}(dx).$$

(26)
Frozen drift approximation

Replace the random terms in the drift by their initial values, yielding:

\[
b^0(s, T_i) = -c_s \lambda(s, T_i) \sum_{l=i+1}^{N} \frac{\delta_l L(0, T_l)}{1 + \delta_l L(0, T_l)} \lambda(s, T_l)
\]

\[- \int_{\mathbb{R}} \left( \left( e^{\lambda(s, T_i)x} - 1 \right) \prod_{l=i+1}^{N} \beta^0(s, x, T_l) - \lambda(s, T_i)x \right) F_{s}^{T^*}(dx). \quad (27)\]

where

\[
\beta^0(t, x, T_l) = \frac{\delta_l L(0, T_l)}{1 + \delta_l L(0, T_l)} \left( e^{\lambda(t, T_l)x} - 1 \right) + 1. \quad (28)
\]

▶ Each LIBOR is driven by a Lévy process. Simulation is very easy.

▶ Yields the same prices as the frozen compensator approximation.
Strong Taylor approximation

**Aim:** develop a method to approximate the random terms, which allows to
1. incorporate and improve the “frozen drift/compensator” approximation,
2. estimate the error of the approximation;

⇝ strong Taylor approximation for perturbed SDEs.

**Definition 1.** A *strong Taylor approximation* of order $n \geq 0$ is a (truncated) power series

$$T_\epsilon^n(W_\epsilon) := \sum_{i=0}^{n} \frac{\epsilon^i}{i!} \frac{\partial^i}{\partial \epsilon^i} \bigg|_{\epsilon=0} W_\epsilon$$

such that

$$\mathbb{E} \left[ |W_\epsilon - T_\epsilon^n(W_\epsilon)| \right] = o(\epsilon^n),$$

holds true as $\epsilon \to 0$. 
Perturbed SDEs

Introduce a process $X^\epsilon$, parameterized by $\epsilon$, with dynamics

$$
\begin{aligned}
\text{d}X^\epsilon_t(T_i) &= \epsilon X^\epsilon_{t-}(T_i) \left( b^\epsilon(t, T_i) \text{d}t + \lambda(t, T_i) \text{d}H_t ight) \\
& \quad + \int_{\mathbb{R}} \left( e^{\lambda(t, T_i)x} - 1 - \lambda(t, T_i)x \right) \mu^H(\text{d}t, \text{d}x),
\end{aligned}
$$

where

$$
\begin{aligned}
b^\epsilon(t, T_i) &= -c_t \lambda(t, T_i) \sum_{l=i+1}^{N} \frac{\delta_l X^\epsilon_{t-}(T_l)}{1 + \delta_l X^\epsilon_{t-}(T_l)} \lambda(t, T_l) \\
& \quad \int_{\mathbb{R}} \left( (e^{\lambda(t, T_i)x} - 1) \prod_{l=i+1}^{N} \beta^\epsilon(t, x, T_l) - \lambda(t, T_i)x \right) F^T_*(\text{d}x),
\end{aligned}
$$

and

$$
\beta^\epsilon(t, x, T_l) = \frac{\delta_l X^\epsilon_{t-}(T_l)}{1 + \delta_l X^\epsilon_{t-}(T_l)} \left( e^{\lambda(t, T_l)x} - 1 \right) + 1.
$$
First order Taylor approximation

The first order strong Taylor approximation of the random variable $X^\epsilon_t(T_i)$ is:

$$\mathbf{T}_\epsilon^1(X^\epsilon_t(T_i)) = X^0_0(T_i) + \epsilon \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} X^\epsilon_t(T_i),$$

where $X^0_0(T_i) \equiv X^0_t(T_i)$ for all $t \in [0, T_i]$. The first variation process of $X^\epsilon_t(T_i)$:

$$d\left(\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} X^\epsilon_t(T_i) \right) := Y_t(T_i) = X^0_0(T_i) \left( b^0(t, T_i)dt + \lambda(t, T_i)dH_t ight)$$

$$+ \int_{\mathbb{R}} \left( e^{\lambda(t,T_i)x} - 1 - \lambda(t, T_i)x \right) \mu^H(dt, dx).$$

Note that $Y(T_i)$ is a Lévy process!
Taylor approximation for the Lévy LIBOR model

Parameterize the LIBOR rate by $\epsilon$; then approximate $L^\epsilon(t, T_i)$ by

$$
d\hat{L}^\epsilon(t, T_i) = \hat{L}^\epsilon(t-, T_i) \left( \hat{b}^\epsilon(t, T_i) dt + \lambda(t, T_i) dH_t ight) + \int_{\mathbb{R}} \left( e^{\lambda(t, T_i)x} - 1 - \lambda(t, T_i)x \right) \mu^H(dt, dx),
$$

(36)

where

$$
\hat{b}^\epsilon(t, T_i) = -c_t \lambda(t, T_i) \sum_{l=i+1}^{N} \frac{\delta_l \left( T^1_{\epsilon} \left( X^\epsilon_{t-}(T_l) \right) \right) + \lambda(t, T_l)}{1 + \delta_l \left( T^1_{\epsilon} \left( X^\epsilon_{t-}(T_l) \right) \right) +} 
\left( e^{\lambda(t, T_i)x} - 1 - \lambda(t, T_i)x \right) \mu^H(dt, dx),
$$

(37)

and

$$
\hat{\beta}^\epsilon(t, x, T_l) = \frac{\delta_l \left( T^1_{\epsilon} \left( X^\epsilon_{t-}(T_l) \right) \right) + \left( e^{\lambda(t, T_i)x} - 1 \right) + 1.}
$$

(38)
Error bounds

Using the Lipschitz property of the map $x \mapsto \frac{\delta x}{1+\delta x}$, we estimate

$$\left| \log \hat{L}^\epsilon(t, T_i) - \log L^\epsilon(t, T_i) \right|$$

$$\leq \int_0^t c_s |\lambda(s, T_i)| \delta^* \sum_{l=i+1}^N \left| X_{s-}^\epsilon(T_l) - T^1_\epsilon(X_{s-}^\epsilon(T_l)) \right| + |\lambda(s, T_l)| ds$$

$$+ \int_0^t \int_{\mathbb{R}} \left| e^{\lambda(s,T_l)x} - 1 \right| \delta^* \sum_{l=i+1}^N \left| X_{s-}^\epsilon(T_l) - T^1_\epsilon(X_{s-}^\epsilon(T_l)) \right| + \left| e^{\lambda(s,T_l)x} - 1 \right| F_{s}^{T^*}(dx) ds$$

$$\leq \int_0^t C(s) \sum_{l=i+1}^N \left| X_{s-}^\epsilon(T_l) - T^1_\epsilon(X_{s-}^\epsilon(T_l)) \right| ds,$$

which is the difference between the actual and approximated drift terms.
Caplet pricing

The price of a caplet, using the relationship between the terminal and the forward measures, can be expressed as

$$
C(K, T_i) = \delta_i B(0, T_{i+1}) \mathbb{E}_{\mathbb{P}_{T_{i+1}}} \left[ (L(T_i, T_i) - K)^+ \right]
$$

$$
= \delta_i B(0, T_*) \mathbb{E}_{\mathbb{P}_{T_*}} \left[ \frac{d\mathbb{P}_{T_{i+1}}}{d\mathbb{P}_{T_*}} |_{\mathcal{F}_{T_i}} (L(T_i, T_i) - K)^+ \right]
$$

$$
= \delta_i B(0, T_*) \mathbb{E}_{\mathbb{P}_{T_*}} \left[ \prod_{l=i+1}^{N} \left( 1 + \delta_i L(T_i, T_l) \right) (L(T_i, T_i) - K)^+ \right], \quad (39)
$$

where the dynamics of $L(\cdot, T_i)$ are described by (22) and (23).

▶ Pricing by an Euler-Maruyama Monte Carlo scheme
Example – numerical illustration

We consider a NIG Lévy process and constant volatilities \( \lambda_i, 1 \leq i \leq N \).
The caplet with maturity \( T_N \) is simply

\[
C(K, T_N) = \delta B(0, T_*) \mathbb{E}_{P_{T_*}}[(L(T_N, T_N) - K)^+],
\]

(40)

where the dynamics of the LIBOR process is

\[
L(t, T_N) = L(0, T_N) \exp \left( \lambda_N H_t - \kappa(\lambda_N)t \right).
\]

(41)

The caplet with maturity \( T_i \) is

\[
C(K, T_i) = \delta B(0, T_*) \mathbb{E}_{P_{T_*}} \left[ \prod_{l=i+1}^{N} \left( 1 + \delta L(T_i, T_l) \right) (L(T_i, T_i) - K)^+ \right],
\]

(42)
where the dynamics of $L(\cdot, T_i)$ are

$$L(t, T_i) = L(0, T_i) \exp \left( \lambda_i H_t - \kappa(\lambda_i)t - \text{Pol}_i \left( \kappa(\lambda_t), \int_0^t \frac{\delta L(s-, T_l)}{1 + \delta L(s-, T_l)} ds \right) \right)$$  \hspace{1cm} (43)$$

where $\text{Pol}_i$ denotes certain polynomials, stemming from (23).

For the strong Taylor approximation, the dynamics are

$$\hat{L}^\epsilon(t, T_i) = \hat{L}^\epsilon(0, T_i) \exp \left( \lambda_i H_t - \kappa(\lambda_i)t - \text{Pol}_i \left( \kappa(\lambda_t), \int_0^t \frac{\delta T^1(X_{s-}^\epsilon(T_l))_+}{1 + \delta T^1(X_{s-}^\epsilon(T_l))_+} ds \right) \right),$$  \hspace{1cm} (44)$$

where

$$T^1(X^\epsilon_t(T_l)) = X^0_0(T_l) + \epsilon Y_t(T_l).$$  \hspace{1cm} (45)$$
and $Y(T_l)$ is a Lévy process with decomposition

$$
Y_t(T_l) = X_0^0(T_l) \left( \lambda_l H_t + \int_0^t \int_\mathbb{R} \left( e^{\lambda_l x} - 1 - \lambda_l x \right) \mu^H(ds, dx) - \text{Pol}_l \left( \kappa(\lambda_k) t, L(0, T_k) \right) \right).
$$

(46)

Accordingly, for the frozen drift approximation the dynamics are

$$
\hat{L}^0(t, T_i) = \hat{L}^0(0, T_i) \exp \left( \lambda_i H_t - \kappa(\lambda_i) t - \text{Pol}_i \left( \kappa(\lambda_i), \frac{\delta L(0, T_l)}{1 + \delta L(0, T_l)} t \right) \right).
$$

(47)

▶ Advantages: significant reduction in computational complexity

▶ Numerical examples: Siopacha & Teichmann (2007) ← diffusion models
Error in bp. for caplet prices: full SDE – frozen drift approximation
Error in bp. for caplet prices: full SDE – strong Taylor approximation
Numerical examples III

Error in bp. for caplet prices: full SDE – frozen drift approximation
Numerical examples IV

Error in bp. for caplet prices: full SDE – strong Taylor approximation
Concluding remarks – Outlook

▶ The frozen drift approximation is quite accurate for caplets $\rightsquigarrow$ fast calibration

▶ The strong Taylor approximation is useful for complex payoffs

▶ The method works for any semimartingale or Lévy-driven SDE ...

▶ Numerical illustrations (work in progress) ...

▶ Application in other settings: multi-currency, defaultable Lévy LIBOR model ...

▶ Preprint forthcoming on my webpage (google papapantoleon).
References