Strong Taylor approximation of SDEs and application to the Lévy LIBOR model

Antonis Papapantoleon

FAM – TU Vienna

Joint work with Maria Siopacha and Friedrich Hubalek

PRisMa 2008: One-Day Workshop on Portfolio Risk Management

TU Vienna, Austria; 29 September 2008

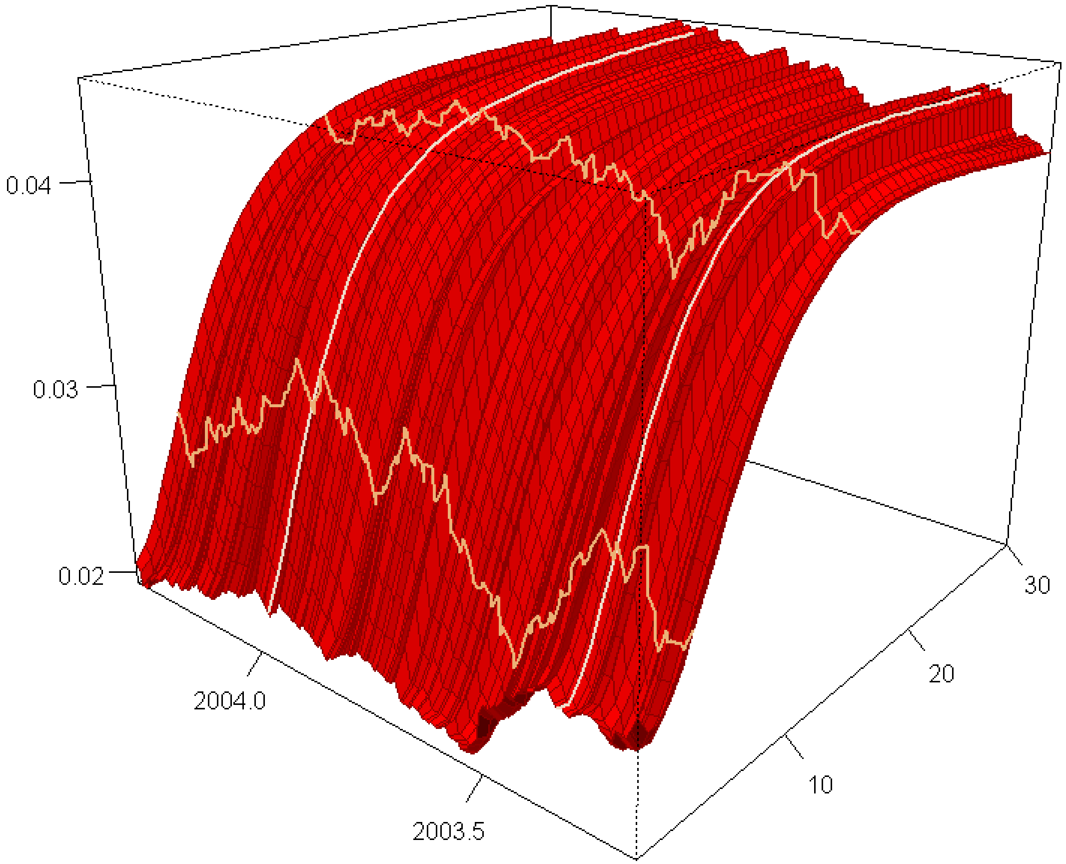
Derivatives markets

According to the BIS Quarterly Review, the notional amounts outstanding for OTC derivatives in billions of US$, were:

	Dec 2006	Jun 2007	Dec 2007
Foreign exchange	40,239	48,620	56,238
Interest rate	291,115	346,937	393,138
Equity-linked	7,488	9,202	8,509
Commodity	7,115	7,567	9,000
Credit default swaps	28,650	42,580	57,894
Unallocated	39,682	61,501	71,225
Total	414,290	516,407	596,004

- Interest rate derivatives are an important part of global financial markets

Interest rates evolution



Evolution of interest rate term structure, 2003 – 2004 (picture: Th. Steiner)

Basic interest rates – Notation

$B(t,T)$ time-t price of a zero coupon bond for maturity T, i.e. $B(T,T)=1$;

$L(t,T)$ time-t forward LIBOR for $[T,T+\delta]$:

$$L(t,T)=\frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)}-1\right)$$

$F(t,T,U)$ time-t forward price for maturities T and U: $F(t,T,U)=\frac{B(t,T)}{B(t,U)}$

Relationships:

$$\boxed{F(t,T,T+\delta)=\frac{B(t,T)}{B(t,T+\delta)}=1+\delta L(t,T)} \qquad (1)$$

Modeling approaches

Short rate models, Forward rate models (HJM), . . .

LIBOR market models: LIBOR rates are modeled directly as

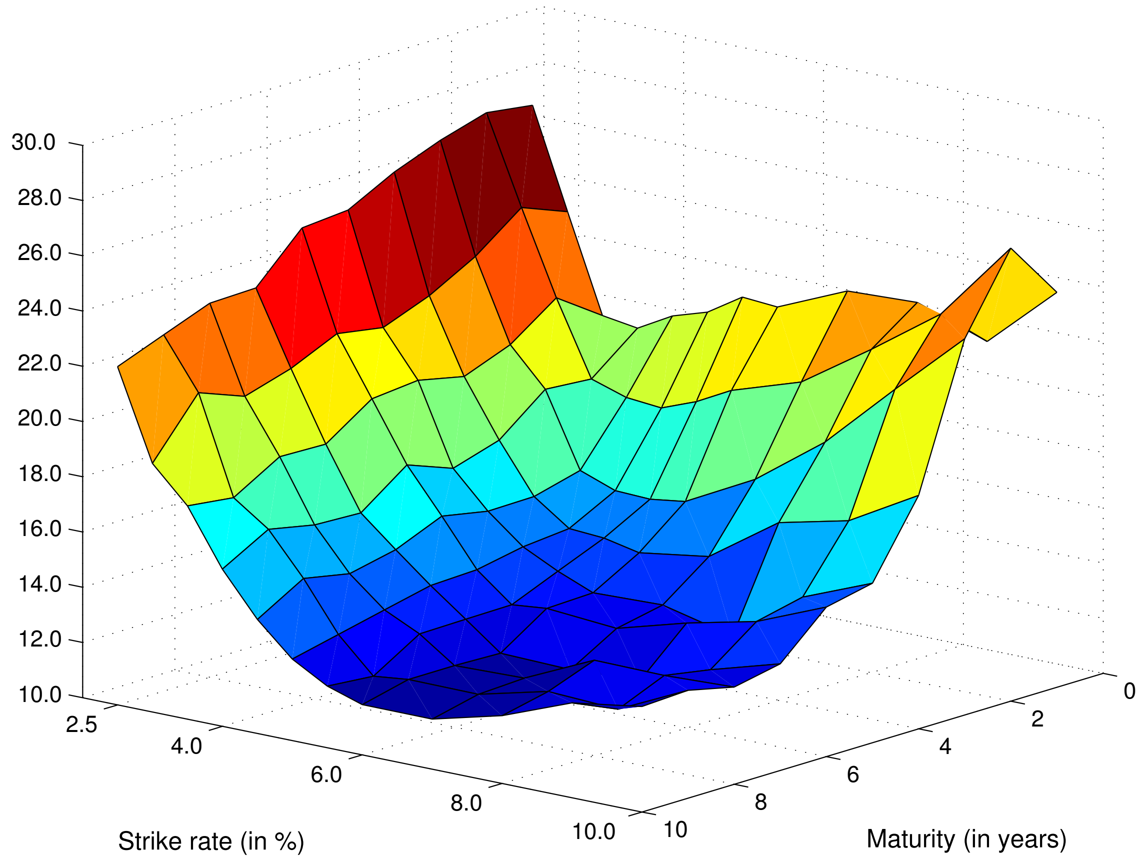
$$L(t,T_k) = L(0,T_k)\mathcal{E}\Big(\int_0^{\cdot} \lambda(s,T_k)\mathrm{d}W_s^{T_{k+1}}\Big)_t$$

for all $T \in \{T_0, T_1, T_2, \ldots, T_{N-1}, T_N\}$, a *discrete* tenor structure. Here $W^{T_{k+1}}$ is a Brownian motion under the *forward martingale measure* $\mathbb{P}_{T_{k+1}}$. Forward measures are related via

$$\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T_k}} = \frac{F(T_k, T_k, T_{k+1})}{F(0, T_k, T_{k+1})}.$$

- **Advantages:** positive LIBOR rates; consistent with market practice i.e. Black 1976. Calibration? Tractability?

Calibration problems



- Caplet implied volatilities are constant neither across strike nor across maturity!

Time-inhomogeneous Lévy processes

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a complete stochastic basis, where $\mathcal{F} = \mathcal{F}_{T_*}$ and $\mathbf{F} = (\mathcal{F}_t)_{t \in [0, T_*]}$.

A time-inhomogeneous Lévy process, is an adapted, càdlàg real-valued stochastic process $H = (H_t)_{0 \leq t \leq T_*}$ with $H_0 = 0$ a.s., such that:

(D1) H has *independent increments*, i.e. $H_t - H_s$ is independent of $\mathcal{F}_s$, $0 \leq s < t \leq T_*$,

(D2) the *law* of H_t is described by the characteristic function

$$\mathbb{E}\left[\mathrm{e}^{iuH_t}\right] = \exp\left(\int_0^t \kappa_s(iu)\mathrm{d}s\right), \tag{2}$$

$$\kappa_s(iu) = iub_s - \frac{u^2 c_s}{2} + \int_{\mathbb{R}} (\mathrm{e}^{iux} - 1 - iux) F_s(\mathrm{d}x), \tag{3}$$

for all $t \in [0, T_*]$. Here $b_s \in \mathbb{R}$, $c_s \in \mathbb{R}_{\geqslant 0}$ and F_s are Lévy measures, $s \in [0, T_*]$.

Assumption ($\mathbb{AC}$). The triplets (b_s, c_s, F_s) satisfy:

$$\int_0^{T_*} \Big(|b_s| + c_s + \int_{\mathbb{R}} (1 \wedge x^2) F_s(\mathrm{d}x) \Big) \mathrm{d}s < \infty.$$

- H is a semimartingale on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. [FTAP applies]

Assumption ($\mathbb{EM}$). There exist $M, \varepsilon > 0$, such that the Lévy measures F_s satisfy:

$$\int_0^{T_*} \int_{\{|x|>1\}} \mathrm{e}^{ux} F_s(\mathrm{d}x) \mathrm{d}s < \infty, \qquad \forall u \in [-(1+\varepsilon)M, (1+\varepsilon)M] =: \mathbb{M}.$$

Moreover, $\int_{\{|x|>1\}} \mathrm{e}^{ux} F_s(\mathrm{d}x) < \infty$ for all $s \in [0, T_*]$ and all $u \in \mathbb{M}$.

- κ_s can be extended to $\mathbb{R} \times i\mathbb{M}$, for all $s \in [0, T_*]$. [Option pricing and Greeks]

Semimartingale semantics

The process H is a *special* semimartingale with canonical decomposition

$$H_t = \int_0^t b_s \mathrm{d}s + \int_0^t \sqrt{c_s}\mathrm{d}W_s + \int_0^t \int_{\mathbb{R}} x(\mu - \nu)(\mathrm{d}s, \mathrm{d}x) \tag{4}$$

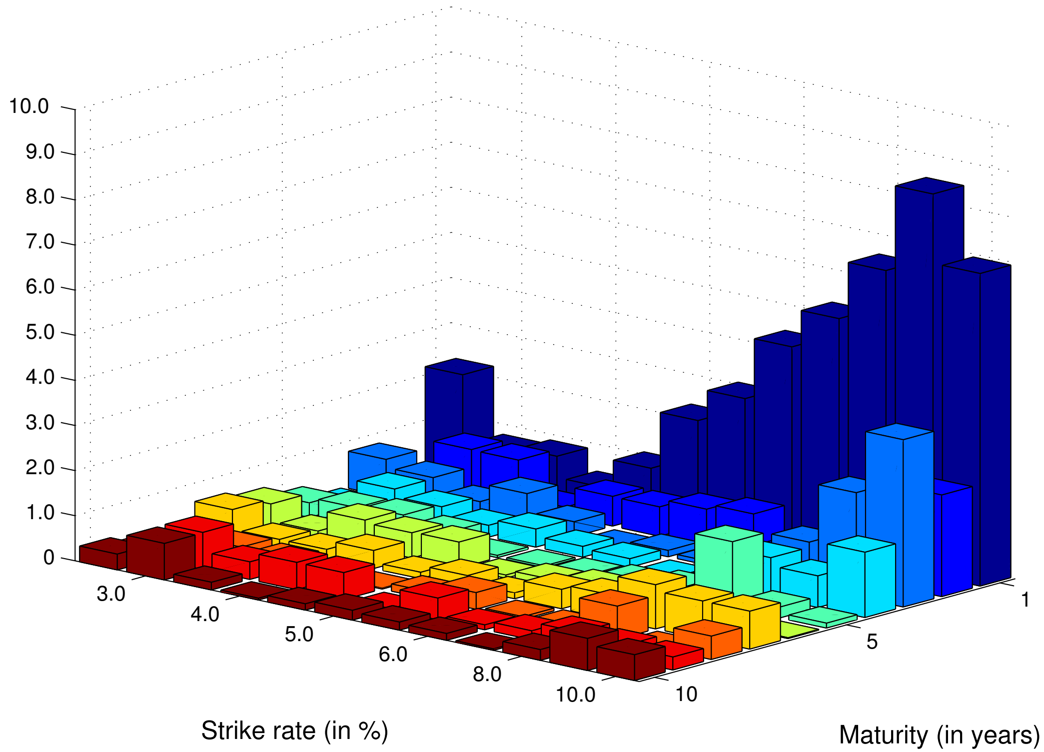
and the triplet of predictable characteristics is (B, C, ν), where

$$B_t = \int_0^t b_s \mathrm{d}s, \qquad C_t = \int_0^t c_s \mathrm{d}s, \qquad \nu([0,t] \times A) = \int_0^t \int_A F_s(\mathrm{d}x)\mathrm{d}s, \tag{5}$$

$A \in \mathcal{B}(\mathbb{R})$. W is a $\mathbb{P}$-Brownian motion, μ is the random measure of jumps of H and ν is the compensator of μ.

- (B, C, ν) completely determines the *distribution* of H.

Calibration results



- Absolute errors in caplet calibration (3-“factor” time-inhomogeneous Lévy process)

The Lévy LIBOR model – construction

Discrete tenor structure: $T_0 < T_1 < T_2 < \cdots < T_N < T_{N+1} = T_*, \quad \delta_i = T_i - T_{i-1}$.

(LR1): For any maturity T_i there exists a bounded, continuous, deterministic volatility function $\lambda(\cdot, T_i) : [0, T_i] \to \mathbb{R}$, corresponding to $L(\cdot, T_i)$. Moreover,

$$\sum_{i=1}^{N} \left|\lambda(s, T_i)\right| \leq M, \qquad \forall s \in [0, T_*].$$

(LR2): Assume a strictly positive and strictly decreasing initial term structure $B(0, T)$, $T \in (0, T_*]$. Consequently

$$L(0, T) = \frac{1}{\delta}\Big(\frac{B(0, T)}{B(0, T + \delta)} - 1\Big) > 0.$$

Let $\mathbb{P}_{T_*}$ be the terminal forward martingale measure. Postulate that the LIBOR with the longest maturity has dynamics

$$L(t, T_N) = L(0, T_N) \exp\Big(\int_0^t b^L(s, T_N) \mathrm{d}s + \int_0^t \lambda(s, T_N) \mathrm{d}H_s \Big), \tag{6}$$

where H is a $\mathbb{P}_{T_*}$-Lévy process, with canonical decomposition

$$H = \int_0^{\cdot} \sqrt{c_s} \mathrm{d}W_s^T + \int_0^{\cdot} \int_{\mathbb{R}} x(\mu^H - \nu^{T_*})(\mathrm{d}s, \mathrm{d}x) \tag{7}$$

and the drift term is

$$b^L(s, T_N) = -\frac{1}{2}\lambda^2(s, T_N) c_s - \int_{\mathbb{R}} \Big(\mathrm{e}^{\lambda(s,T_N)x} - 1 - \lambda(s, T_N)x \Big) F_s^{T_*}(\mathrm{d}x). \tag{8}$$

Equivalently,

$$L(t, T_N) = L(0, T_N)\mathcal{E}\Big(Z(T_N)\Big)_t\,, \tag{9}$$

where $Z(T_N)$ is the exponential transform of $\int_0^{\cdot} \lambda(s, T_N)\mathrm{d}H_s$, i.e.

$$Z(T_N) = \int\limits_0^{\cdot} \lambda(s, T_N)\mathrm{d}H_s + \int\limits_0^{\cdot}\int\limits_{\mathbb{R}} \big(\mathrm{e}^{\lambda(s,T_N)x} - 1 - \lambda(s, T_N)x\big)\mu^H(\mathrm{d}s, \mathrm{d}x). \tag{10}$$

Therefore, $L(\cdot, T_N)$ satisfies the SDE

$$\mathrm{d}L(t, T_N) = L(t-, T_N)\mathrm{d}Z_t(T_N), \tag{11}$$

with initial condition $L(0, T_1) = \frac{1}{\delta}\Big(\frac{B(0,T_N)}{B(0,T)} - 1\Big)$.

Define the forward measure $\mathbb{P}_{T_N}$ via

$$\frac{\mathrm{d}\mathbb{P}_{T_N}}{\mathrm{d}\mathbb{P}_{T_*}} = \frac{F(T_N, T_N, T_*)}{F(0, T_N, T_*)} = \frac{1 + \delta L(T_N, T_N)}{1 + \delta L(0, T_N)}. \tag{12}$$

Then, the dynamics of the forward rate is

$$\begin{aligned} \mathrm{d}F(t, T_N, T_*) &= \delta \mathrm{d}L(t, T_N) \\ &= F(t-, T_N, T)\Bigg(\frac{\delta L(t-, T_N)}{1 + \delta L(t-, T_N)}\lambda(t, T_N)\mathrm{d}H_t \\ &\qquad + \frac{\delta L(t-, T_N)}{1 + \delta L(t-, T_N)}\int_{\mathbb{R}} (\mathrm{e}^{\lambda(t, T_N)x} - 1 - \lambda(s, T_N)x)\mu^H(\mathrm{d}t, \mathrm{d}x)\Bigg). \end{aligned} \tag{13}$$

Now, the next LIBOR rate has dynamics

$$L(t,T_{N-1}) = L(0,T_{N-1})\exp\Big(\int_0^t b^L(s,T_{N-1})\mathrm{d}s + \int_0^t \lambda(s,T_{N-1})\mathrm{d}H_s^{T_N}\Big), \qquad (14)$$

where H^{T_N} is a semimartingale with canonical decomposition

$$H^{T_N} = \int_0^{\cdot} \sqrt{c_s}\mathrm{d}W_s^{T_N} + \int_0^{\cdot}\int_{\mathbb{R}} x(\mu^H - \nu^{T_N})(\mathrm{d}s,\mathrm{d}x); \qquad (15)$$

while $b^L(\cdot,T_{N-1})$ ensures martingality. Here the $\mathbb{P}_{T_N}$-Brownian motion is

$$W^{T_N} = W^{T_*} - \int_0^{\cdot} \frac{\delta L(s-,T_N)}{1+\delta L(s-,T_N)}\lambda(s,T_N)\sqrt{c_t}\mathrm{d}s, \qquad (16)$$

and the $\mathbb{P}_{T_N}$-compensator of μ^H is

$$\nu^{T_N}(\mathrm{d}t,\mathrm{d}x) = \Big(\frac{\delta L(t-,T_N)}{1+\delta L(t-,T_N)}(\mathrm{e}^{\lambda(t,T_N)x}-1)+1\Big)\cdot\nu^{T_*}(\mathrm{d}t,\mathrm{d}x). \qquad (17)$$

The construction proceeds inductively; the dynamics of the LIBOR $L(\cdot, T_i)$ under the forward measure $\mathbb{P}_{T_{i+1}}$, is

$$L(t, T_i) = L(0, T_i) \exp\Big(\int_0^t b^L(s, T_i) \mathrm{d}s + \int_0^t \lambda(s, T_i) \mathrm{d}H_s^{T_{i+1}} \Big), \tag{18}$$

where $b^L(\cdot, T_i)$ ensures the martingality of the rate, and

$$H^{T_{i+1}} = \int_0^{\cdot} \sqrt{c_s} \mathrm{d}W_s^{T_{i+1}} + \int_0^{\cdot} \int_{\mathbb{R}} x(\mu^H - \nu^{T_{i+1}})(\mathrm{d}s, \mathrm{d}x). \tag{19}$$

$W^{T_{i+1}}$ is a $\mathbb{P}_{T_{i+1}}$-Brownian motion and the $\mathbb{P}_{T_{i+1}}$-compensator of μ^H is

$$\nu^{T_{i+1}}(\mathrm{d}s, \mathrm{d}x) = \prod_{l=i+1}^{N} \underbrace{\frac{\delta L(s-, T_l)}{1 + \delta L(s-, T_l)} \Big(\Big(\mathrm{e}^{\lambda(s, T_l) x} - 1 \Big) + 1 \Big)}_{:= \beta(s, x, T_l)} \nu^T(\mathrm{d}s, \mathrm{d}x). \tag{20}$$

- Consequence: $H^{T_{i+1}}$ is not a time-inhomogeneous Lévy process (unless $i = N$).

Frozen compensator approximation

Idea: replace the random terms by their deterministic initial values

$$\frac{\delta L(t-,T_l)}{1+\delta L(t-,T_l)} \approx \frac{\delta L(0,T_l)}{1+\delta L(0,T_l)}$$

in the compensator $\nu^{T_{i+1}}$ of μ^H under the forward martingale measure $\mathbb{P}_{T_{i+1}}$; i.e

$$\nu^{0,T_{i+1}}(\mathrm{d}s,\mathrm{d}x) = \prod_{l=i+1}^{N} \frac{\delta L(0,T_l)}{1+\delta L(0,T_l)} \left(\left(\mathrm{e}^{\lambda(s,T_l)x} - 1\right) + 1\right) \nu^{T_*}(\mathrm{d}s,\mathrm{d}x). \tag{21}$$

- $H^{0,T_{i+1}}$ is a time-inhomogeneous Lévy process.
- Pricing is possible (but hard). Simulation is *very* hard ...
- What is the error? The error is compounded ...

Terminal measure dynamics

We lift all LIBOR rates to the terminal measure $\mathbb{P}_{T_*}$:

$$L(t,T_i) = L(0,T_i)\exp\left(\int_0^t \bar{b}(s,T_i)\mathrm{d}s + \int_0^t \lambda(s,T_i)\mathrm{d}H_s\right), \tag{22}$$

where H is the *driving* time-inhomogeneous Lévy process, and

$$\bar{b}(s,T_i) = -\frac{1}{2}\lambda^2(s,T_i)c_s - c_s\lambda(s,T_i)\sum_{l=i+1}^{N}\frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}\lambda(s,T_l)$$
$$-\int_{\mathbb{R}}\left(\left(\mathrm{e}^{\lambda(s,T_i)x}-1\right)\prod_{l=i+1}^{N}\beta(s,x,T_l)-\lambda(s,T_i)x\right)F_s^{T_*}(\mathrm{d}x). \tag{23}$$

This can also be expressed as a stochastic exponential.

Under the terminal measure, the dynamics of $L(\cdot, T_i)$ satisfy the SDE

$$\mathrm{d}L(t, T_i) = L(t-, T_i)\mathrm{d}Z_t \tag{24}$$

where $Z = (Z_t)_{0 \le t \le T}$ is a semimartingale with canonical decomposition

$$\begin{aligned} Z = \int_0^{\cdot} b(s, T_i)\mathrm{d}s + \int_0^{\cdot} \lambda(s, T_i)\mathrm{d}H_s \\ + \int_0^{\cdot} \int_{\mathbb{R}} (\mathrm{e}^{\lambda(s,T_i)x} - 1 - \lambda(s, T_i)x)\mu^H(\mathrm{d}s, \mathrm{d}x) \end{aligned} \tag{25}$$

and the (random) drift term is

$$\begin{aligned} b(s, T_i) = -c_s\lambda(s, T_i) \sum_{l=i+1}^{N} \frac{\delta_l L(s-, T_l)}{1 + \delta_l L(s-, T_l)} \lambda(s, T_l) \\ - \int_{\mathbb{R}} \left(\left(\mathrm{e}^{\lambda(s,T_i)x} - 1 \right) \prod_{l=i+1}^{N} \beta(s, x, T_l) - \lambda(s, T_i)x \right) F_s^{T_*}(\mathrm{d}x). \end{aligned} \tag{26}$$

Frozen drift approximation

Replace the random terms in the drift by their initial values, yielding:

$$b^0(s,T_i) = -c_s\lambda(s,T_i)\sum_{l=i+1}^{N}\frac{\delta_l L(0,T_l)}{1+\delta_l L(0,T_l)}\lambda(s,T_l) - \int_{\mathbb{R}}\left(\left(\mathrm{e}^{\lambda(s,T_i)x}-1\right)\prod_{l=i+1}^{N}\beta^0(s,x,T_l) - \lambda(s,T_i)x\right)F_s^{T_*}(\mathrm{d}x). \tag{27}$$

where

$$\beta^0(t,x,T_l) = \frac{\delta_l L(0,T_l)}{1+\delta_l L(0,T_l)}\left(\mathrm{e}^{\lambda(t,T_l)x}-1\right)+1. \tag{28}$$

- Each LIBOR is driven by a Lévy process. Simulation is *very easy*.
- Yields the same prices as the *frozen compensator* approximation.

Strong Taylor approximation

Aim: develop a method to approximate the random terms, which allows to

1. incorporate and improve the "frozen drift/compensator" approximation,
2. estimate the error of the approximation;

$\rightsquigarrow$ strong Taylor approximation for perturbed SDEs.

Definition 1. A *strong Taylor approximation* of order $n \geq 0$ is a (truncated) power series

$$\mathbf{T}_\epsilon^n(W_\epsilon) := \sum_{i=0}^{n} \frac{\epsilon^i}{i!} \frac{\partial^i}{\partial \epsilon^i}\Big|_{\epsilon=0} W_\epsilon \tag{29}$$

such that

$$\mathbb{E}\left[|W_\epsilon - \mathbf{T}_\epsilon^n(W_\epsilon)|\right] = o(\epsilon^n), \tag{30}$$

holds true as $\epsilon \to 0$.

Perturbed SDEs

Introduce a process X^ϵ, parameterized by ϵ, with dynamics

$$\mathrm{d}X_t^\epsilon(T_i) = \epsilon X_{t-}^\epsilon(T_i)\Big(b^\epsilon(t,T_i)\mathrm{d}t + \lambda(t,T_i)\mathrm{d}H_t + \int_{\mathbb{R}} \Big(\mathrm{e}^{\lambda(t,T_i)x} - 1 - \lambda(t,T_i)x\Big)\mu^H(\mathrm{d}t,\mathrm{d}x)\Big), \quad (31)$$

where

$$b^\epsilon(t,T_i) = -c_t\lambda(t,T_i)\sum_{l=i+1}^{N}\frac{\delta_l X_{t-}^\epsilon(T_l)}{1+\delta_l X_{t-}^\epsilon(T_l)}\lambda(t,T_l) - \int_{\mathbb{R}}\Big(\Big(\mathrm{e}^{\lambda(t,T_i)x} - 1\Big)\prod_{l=i+1}^{N}\beta^\epsilon(t,x,T_l) - \lambda(t,T_i)x\Big)F_t^{T_*}(\mathrm{d}x), \quad (32)$$

and

$$\beta^\epsilon(t,x,T_l) = \frac{\delta_l X_{t-}^\epsilon(T_l)}{1+\delta_l X_{t-}^\epsilon(T_l)}\Big(\mathrm{e}^{\lambda(t,T_l)x} - 1\Big) + 1. \quad (33)$$

First order Taylor approximation

The first order strong Taylor approximation of the random variable $X_t^\epsilon(T_i)$ is :

$$\mathbf{T}_\epsilon^1\big(X_t^\epsilon(T_i)\big) = X_0^0(T_i) + \epsilon\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} X_t^\epsilon(T_i), \tag{34}$$

where $X_0^0(T_i) \equiv X_t^0(T_i)$ for all $t \in [0, T_i]$. The first variation process of $X^\epsilon(T_i)$:

$$d\Big(\underbrace{\frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0} X_t^\epsilon(T_i)}_{:=Y_t(T_i)}\Big) = X_0^0(T_i)\Big(b^0(t,T_i)\mathrm{d}t + \lambda(t,T_i)\mathrm{d}H_t + \int_{\mathbb{R}} (\mathrm{e}^{\lambda(t,T_i)x} - 1 - \lambda(t,T_i)x)\mu^H(\mathrm{d}t,\mathrm{d}x)\Big). \tag{35}$$

- Note that $Y(T_i)$ is a Lévy process!

Taylor approximation for the Lévy LIBOR model

Parameterize the LIBOR rate by ϵ; then approximate $L^{\epsilon}(t,T_i)$ by

$$\mathrm{d}\widehat{L}^{\epsilon}(t,T_i) = \widehat{L}^{\epsilon}(t-,T_i)\Big(\widehat{b}^{\epsilon}(t,T_i)\mathrm{d}t + \lambda(t,T_i)\mathrm{d}H_t + \int_{\mathbb{R}}(\mathrm{e}^{\lambda(t,T_i)x} - 1 - \lambda(t,T_i)x)\mu^H(\mathrm{d}t,\mathrm{d}x)\Big), \tag{36}$$

where

$$\widehat{b}^{\epsilon}(t,T_i) = -c_t\lambda(t,T_i)\sum_{l=i+1}^{N}\frac{\delta_l\big(\mathbf{T}_{\epsilon}^{1}\big(X_{t-}^{\epsilon}(T_l)\big)\big)_+}{1+\delta_l\big(\mathbf{T}_{\epsilon}^{1}\big(X_{t-}^{\epsilon}(T_l)\big)\big)_+}\lambda(t,T_l) - \int_{\mathbb{R}}\bigg(\Big(\mathrm{e}^{\lambda(t,T_i)x}-1\Big)\prod_{l=i+1}^{N}\widehat{\beta}^{\epsilon}(t,x,T_l) - \lambda(t,T_i)x\bigg)F_t^{T_*}(\mathrm{d}x), \tag{37}$$

and

$$\widehat{\beta}^{\epsilon}(t,x,T_l) = \frac{\delta_l\big(\mathbf{T}_{\epsilon}^{1}\big(X_{t-}^{\epsilon}(T_l)\big)\big)_+}{1+\delta_l\big(\mathbf{T}_{\epsilon}^{1}\big(X_{t-}^{\epsilon}(T_l)\big)\big)_+}\Big(\mathrm{e}^{\lambda(t,T_l)x}-1\Big)+1. \tag{38}$$

Error bounds

Using the Lipschitz property of the map $x \mapsto \frac{\delta x}{1+\delta x}$, we estimate

$$
\begin{aligned}
& |\log \widehat{L}^\epsilon(t,T_i) - \log L^\epsilon(t,T_i)| \\
& \quad \leq \int_0^t c_s |\lambda(s,T_i)| \delta^* \sum_{l=i+1}^N \Big| X^\epsilon_{s-}(T_l) - \mathbf{T}^1_\epsilon\big(X^\epsilon_{s-}(T_l)\big)_+ \Big| |\lambda(s,T_l)| \mathrm{d}s \\
& \quad + \int_0^t \int_{\mathbb{R}} \Big| \mathrm{e}^{\lambda(s,T_i)x} - 1 \Big| \delta^* \sum_{l=i+1}^N \Big| X^\epsilon_{s-}(T_l) - \mathbf{T}^1_\epsilon\big(X^\epsilon_{s-}(T_l)\big)_+ \Big| \Big| \mathrm{e}^{\lambda(s,T_l)x} - 1 \Big| F_s^{T_*}(\mathrm{d}x)\mathrm{d}s \\
& \quad \leq \int_0^t \mathcal{C}(s) \sum_{l=i+1}^N \Big| X^\epsilon_{s-}(T_l) - \mathbf{T}^1_\epsilon\big(X^\epsilon_{s-}(T_l)\big)_+ \Big| \mathrm{d}s,
\end{aligned}
$$

which is the difference between the actual and approximated drift terms.

Caplet pricing

The price of a caplet, using the relationship between the terminal and the forward measures, can be expressed as

$$
\begin{aligned}
\mathbb{C}(K,T_i) &= \delta_i B(0,T_{i+1})\mathbb{E}_{\mathbb{P}_{T_{i+1}}}[(L(T_i,T_i)-K)^+] \\
&= \delta_i B(0,T_*)\mathbb{E}_{\mathbb{P}_{T_*}}\Big[\frac{\mathrm{d}\mathbb{P}_{T_{i+1}}}{\mathrm{d}\mathbb{P}_{T_*}}\Big|_{\mathcal{F}_{T_i}}(L(T_i,T_i)-K)^+\Big] \\
&= \delta_i B(0,T_*)\mathbb{E}_{\mathbb{P}_{T_*}}\Big[\prod_{l=i+1}^{N}\Big(1+\delta_l L(T_i,T_l)\Big)(L(T_i,T_i)-K)^+\Big], \qquad (39)
\end{aligned}
$$

where the dynamics of $L(\cdot,T_l)$ are described by (22) and (23).

- Pricing by an Euler-Maruyama Monte Carlo scheme

Example – numerical illustration

We consider a NIG Lévy process and constant volatilities λ_i, $1 \leq i \leq N$.
The caplet with maturity T_N is simply

$$\mathbb{C}(K, T_N) = \delta B(0, T_*) \mathbb{E}_{\mathbb{P}_{T_*}}[(L(T_N, T_N) - K)^+], \tag{40}$$

where the dynamics of the LIBOR process is

$$L(t, T_N) = L(0, T_N) \exp\Big(\lambda_N H_t - \kappa(\lambda_N) t\Big). \tag{41}$$

The caplet with maturity T_i is

$$\mathbb{C}(K, T_i) = \delta B(0, T_*) \mathbb{E}_{\mathbb{P}_{T_*}}\Big[\prod_{l=i+1}^{N} \big(1 + \delta L(T_i, T_l)\big)(L(T_i, T_i) - K)^+\Big], \tag{42}$$

where the dynamics of $L(\cdot, T_i)$ are

$$L(t,T_i) = L(0,T_i)\exp\left(\lambda_i H_t - \kappa(\lambda_i)t - \mathsf{Pol}_i\Big(\kappa(\lambda_l), \int\limits_0^t \frac{\delta L(s-,T_l)}{1+\delta L(s-,T_l)}\mathrm{d}s\Big)\right) \quad (43)$$

where Pol_i denotes certain polynomials, stemming from (23).

For the strong Taylor approximation, the dynamics are

$$\widehat{L}^\epsilon(t,T_i) = \widehat{L}^\epsilon(0,T_i)\exp\left(\lambda_i H_t - \kappa(\lambda_i)t - \mathsf{Pol}_i\Big(\kappa(\lambda_l), \int\limits_0^t \frac{\delta \mathbf{T}^1(X^\epsilon_{s-}(T_l))_+}{1+\delta \mathbf{T}^1(X^\epsilon_{s-}(T_l))_+}\mathrm{d}s\Big)\right), \quad (44)$$

where

$$\mathbf{T}^1(X^\epsilon_t(T_l)) = X^0_0(T_l) + \epsilon Y_t(T_l), \quad (45)$$

and $Y(T_l)$ is a Lévy process with decomposition

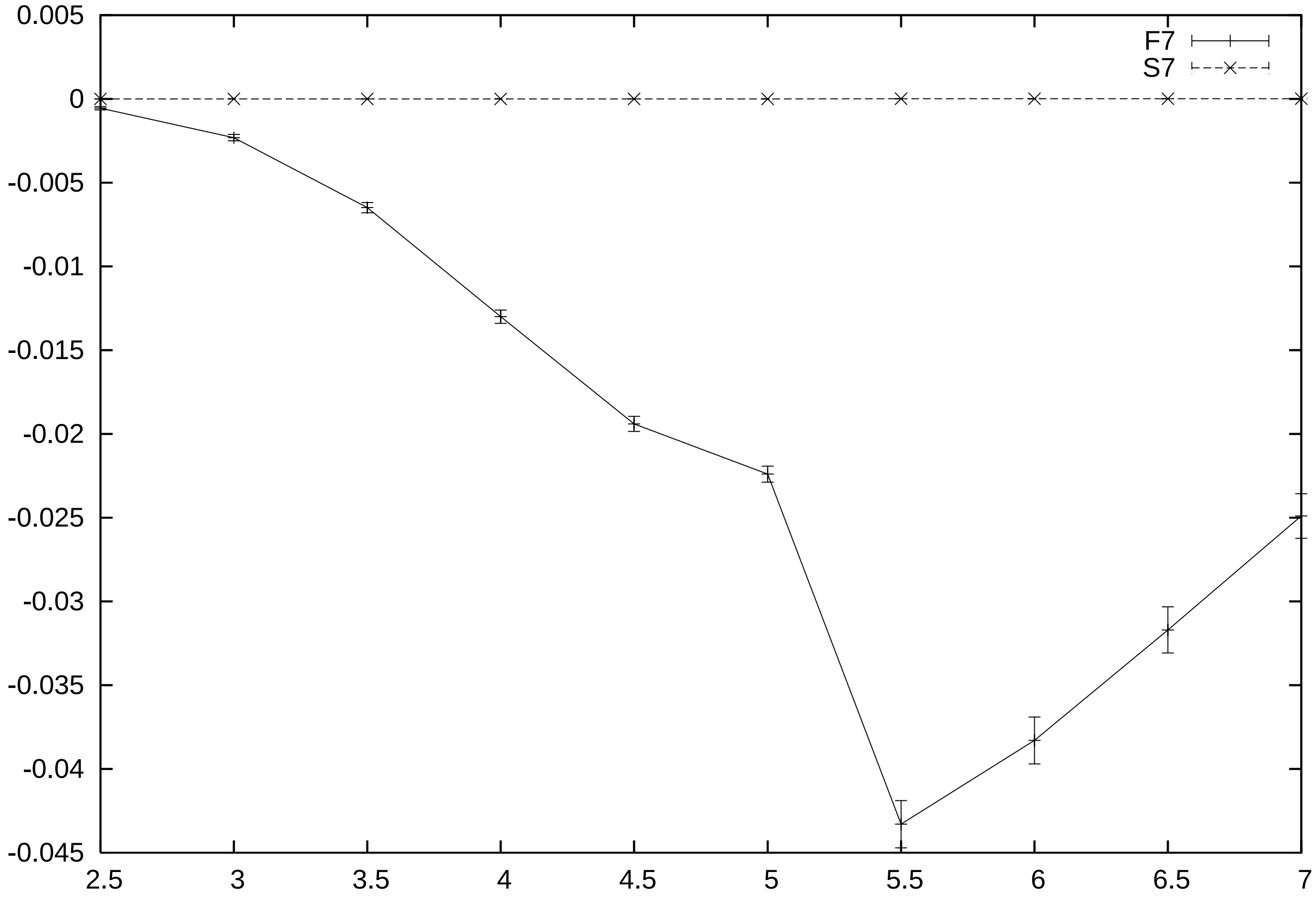
$$Y_t(T_l) = X_0^0(T_l)\left(\lambda_l H_t + \int_0^t \int_{\mathbb{R}} (\mathrm{e}^{\lambda_l x} - 1 - \lambda_l x)\mu^H(\mathrm{d}s, \mathrm{d}x) - \mathsf{Pol}_l\Big(\kappa(\lambda_k)t, L(0, T_k)\Big)\right). \tag{46}$$

Accordingly, for the frozen drift approximation the dynamics are

$$\widehat{L}^0(t, T_i) = \widehat{L}^0(0, T_i) \exp\left(\lambda_i H_t - \kappa(\lambda_i)t - \mathsf{Pol}_i\Big(\kappa(\lambda_l), \frac{\delta L(0, T_l)}{1 + \delta L(0, T_l)} t\Big)\right). \tag{47}$$

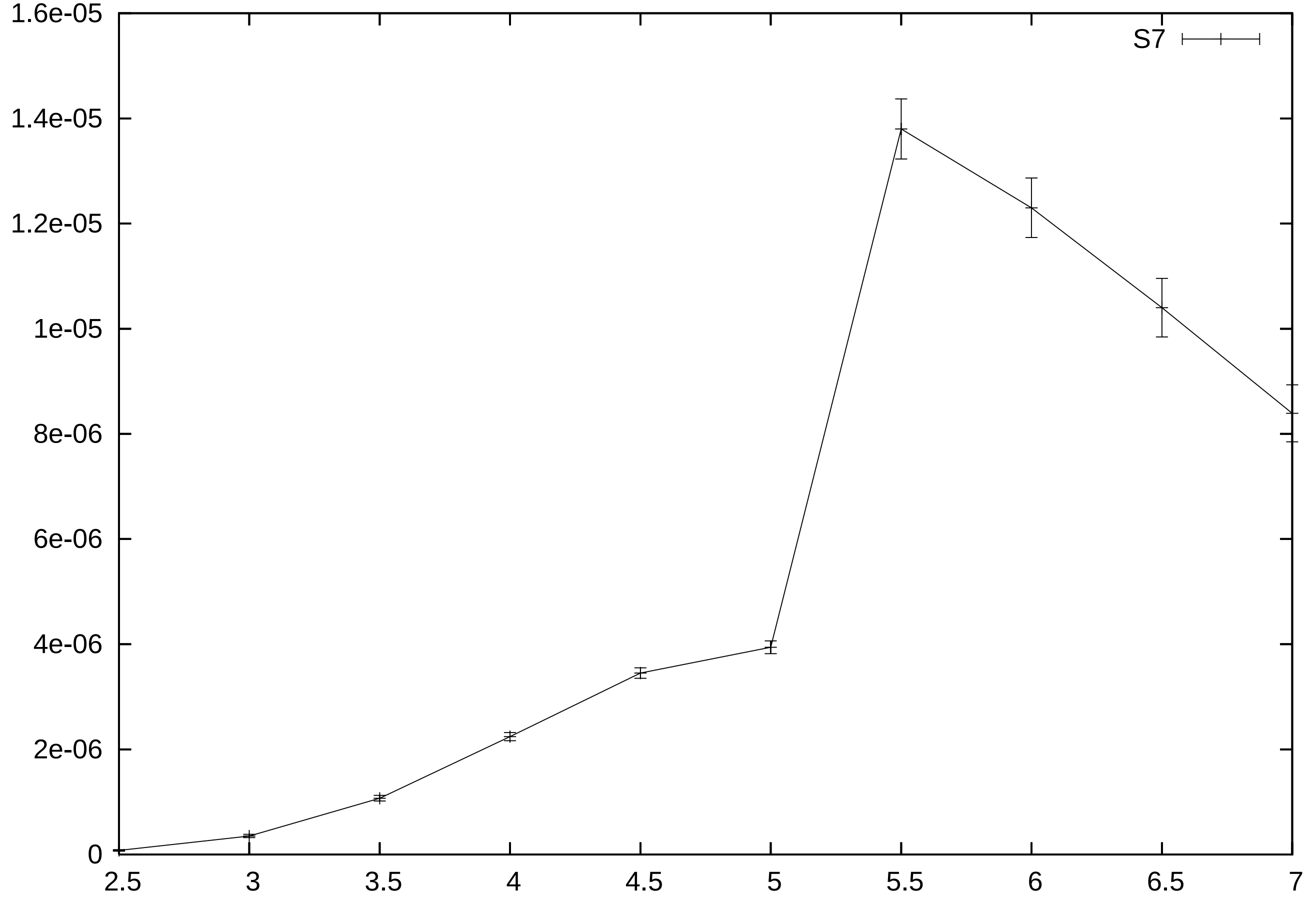
- Advantages: significant reduction in computational complexity
- Numerical examples: Siopacha & Teichmann (2007) $\hookrightarrow$ diffusion models

Numerical examples I



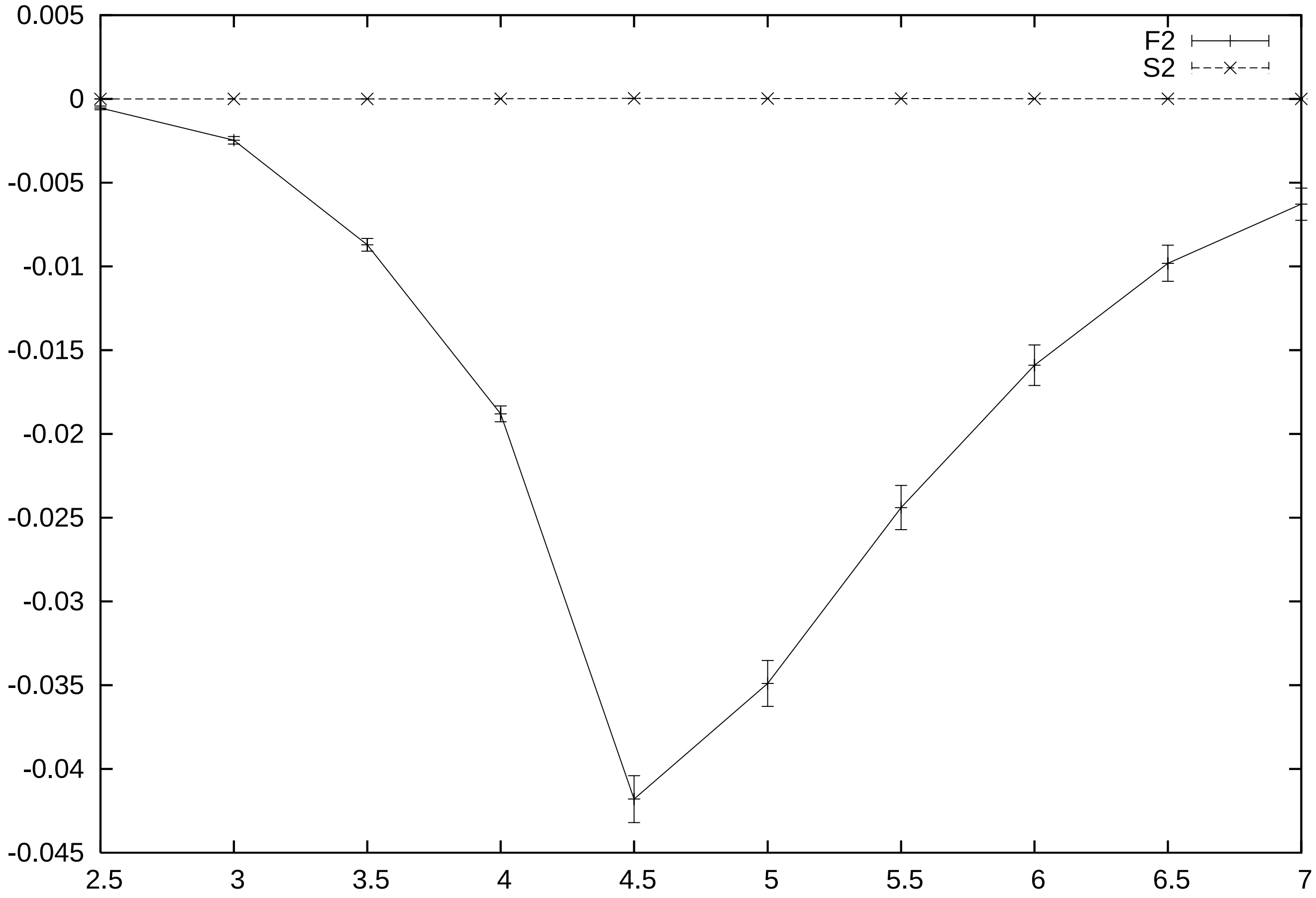
Error in bp. for caplet prices: full SDE – frozen drift approximation

Numerical examples II



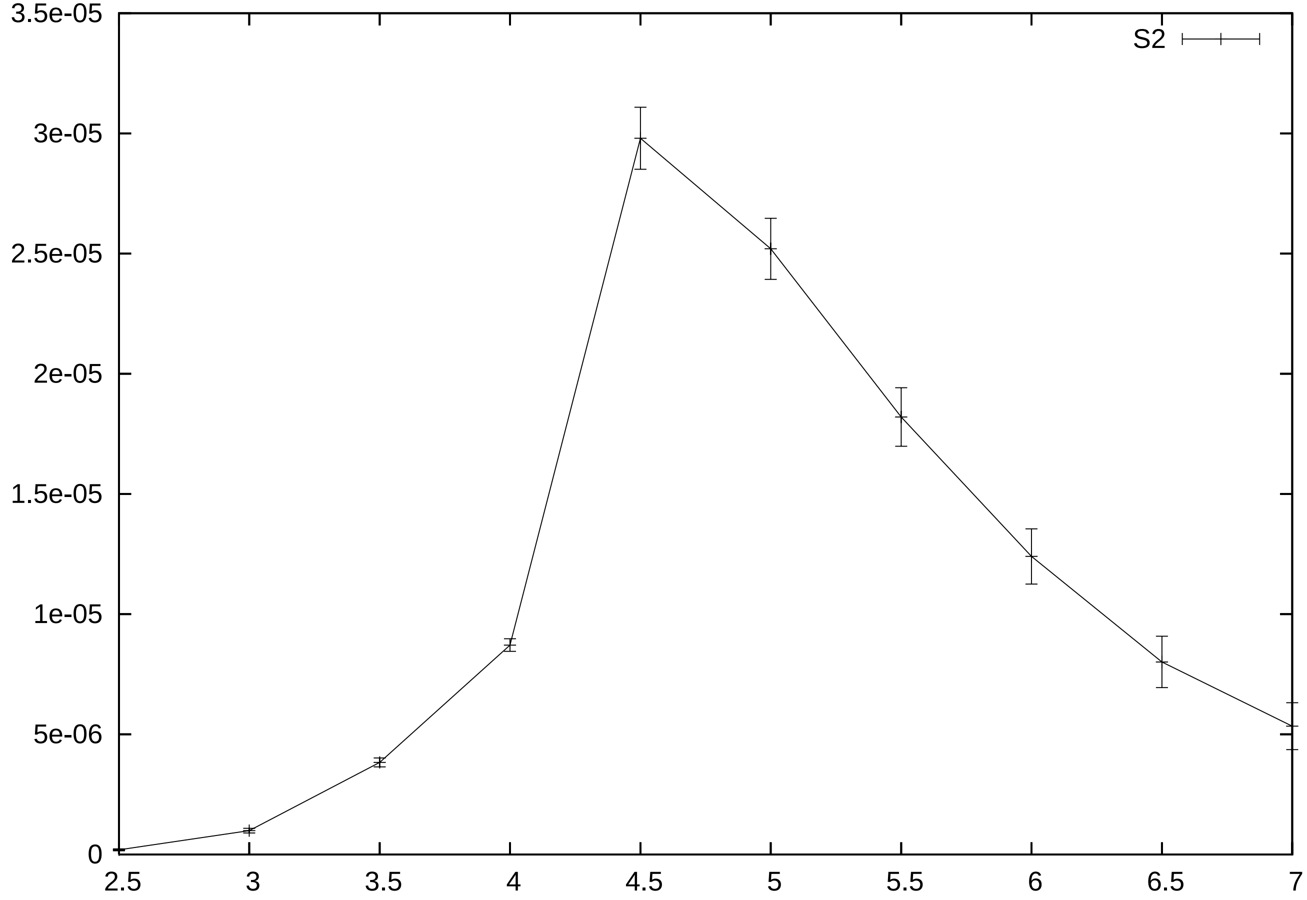
Error in bp. for caplet prices: full SDE – strong Taylor approximation

Numerical examples III



Error in bp. for caplet prices: full SDE – frozen drift approximation

Numerical examples IV



Error in bp. for caplet prices: full SDE – strong Taylor approximation

Concluding remarks – Outlook

- The frozen drift approximation is quite accurate for caplets $\rightsquigarrow$ fast calibration
- The strong Taylor approximation is useful for complex payoffs
- The method works for any semimartingale or Lévy-driven SDE ...
- Numerical illustrations (work in progress) ...
- Application in other settings: multi-currency, defaultable Lévy LIBOR model ...

- Preprint forthcoming on my webpage (google papapantoleon).

References

- F. Black (1976). The pricing of commodity contracts. *J. Financ. Econ. 3*, 167-179.
- A. Brace, D. Gątarek, M. Musiela (1997). The market model of interest rate dynamics. *Math. Finance 7*, 127-155.
- D. Duffie, D. Filipovic, W. Schachermayer (2003). Affine processes and applications in finance. *Ann. Appl. Probab. 13*, 984-1053.
- E. Eberlein, F. Özkan (2005). The Lévy LIBOR model. *Finance Stoch. 9*, 327-348.
- J. Jacod, A.N. Shiryaev (2003). *Limit theorems for stochastic processes* (2nd. ed.). Springer.
- F. Jamshidian (1997). LIBOR and swap market models and measures. *Finance Stoch. 1*, 293-330.
- J. Kallsen, A. N. Shiryaev (2002). The cumulant process and Esscher's change of measure. *Finance Stoch. 6*, 397-428.
- W. Kluge (2005). *Time-inhomogeneous Lévy processes in interest rate and credit risk models*. Ph.D. thesis, University of Freiburg.
- K. R. Miltersen, K. Sandmann, D. Sondermann (1997). Closed form solutions for term structure derivatives with log-normal interest rates. *J. Finance 52*, 409-430.
- M. Musiela, M. Rutkowski (1997). Continuous-time term structure models: forward measure approach. *Finance Stoch. 1*, 261-291.
- K. Sandmann, D. Sondermann, K. R. Miltersen (1995). Closed form term structure derivatives in a Heath-Jarrow-Morton model with lognormal annually compounded interest rates. In *Proceedings of the Seventh Annual European Futures Research Symposium Bonn*, pp. 145-165.
- M. Siopacha, J. Teichmann (2007). Weak and strong Taylor methods for numerical solutions of stochastic differential equations. Preprint `arXiv/0704.0745`.