DYNAMIC CDO TERM STRUCTURE MODELLING

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PRisMa 2008
Workshop on Portfolio Risk Management
TU Vienna, 29 September 2008

Overview

1. **Collateralized Debt Obligations (CDOs)**

2. (T, x)-Bonds

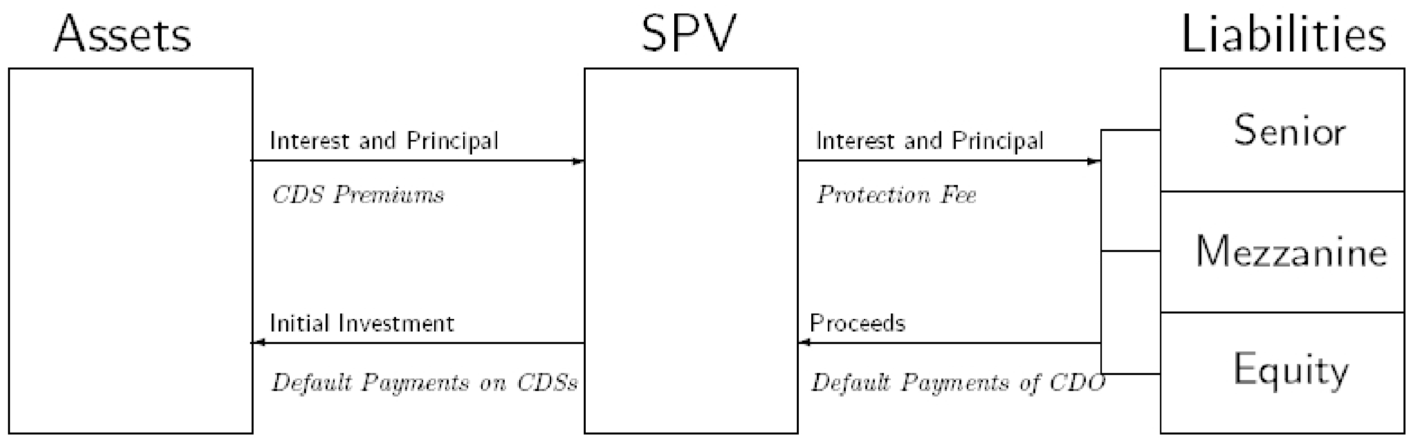
3. Arbitrage-free Term Structure Movements

4. Doubly Stochastic Framework

Collateralized Debt Obligations (CDOs)

- most important type of portfolio credit derivative
- security backed by pool of reference entities (**assets**): bonds, loans, protection seller position in single name CDS, etc.
- assets sold to **special-purpose vehicle (SPV)**
- SPV issues notes on **CDO tranches** (**liabilities**)
- important reference indices: ITraxx Europe, CDX (USA),

Basic Structure of a CDO



Payments in a CDO structure.

Payments corresponding to synthetic CDOs are in *italics*.

Literature

Bennani (05): "The forward loss model: A dynamic term structure approach for the pricing of portfolio credit derivatives"

Cont and Minca (08), "Recovering portfolio default intensities implied by CDO quotes"

Ehlers and Schönbucher (06), "Pricing Interest Rate-Sensitive Credit Portfolio Derivatives"

Ehlers and Schönbucher (06), "Background Filtrations and Canonical Loss Processes for Top-Down Models of Portfolio Credit Risk"

Filipović, Overbeck and Schmidt (08), "Dynamic CDO Term Structure Modelling"

Schönbucher (05), "Portfolio losses and the term structure of loss transition rates: A new methodology for the pricing of portfolio credit derivatives"

Sidenius, Piterbarg and Andersen (IJTAF 08), "A new framework for dynamic credit portfolio loss modelling"

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(T, x)-Bonds

- $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{Q})$, $\mathbb{Q}$ risk-neutral measure

- CDO pool of credits normalized to 1.

- **Loss process** $L_t = \sum_{s \le t} \Delta L_s$ $[0, 1]$-valued increasing MPP with abs. continuous compensator $\nu(t, dx)\, dt$.

- **(T, x)-bond** pays $1_{\{L_T \le x\}}$ at maturity T, $x \in [0, 1]$.
  Its price $P(t, T, x)$ at $t \le T$ is decreasing in T, increasing in x.
  Note: $P(t, T) = P(t, T, 1)$ is risk-free zero-coupon bond.

Default Times of the (T, x)-Bonds

Lemma 1: For any $x \in [0,1]$, the process $1_{\{L_t \le x\}}$ has **intensity**

$$\lambda(t,x) = \nu(t,(x - L_t, 1]).$$

That is,

$$M_t^x = 1_{\{L_t \le x\}} + \int_0^t 1_{\{L_s \le x\}} \lambda(s,x)\, ds \quad \text{is a martingale.}$$

Conversely, $\lambda(t,x)$ uniquely determines $\nu(t,dx)$ via

$$\nu(t,(0,x]) = \lambda(t,L_t) - \lambda(t,L_t + x), \quad x \in [0,1].$$

Proof. $F(L_t) - \int_0^t \int_0^1 (F(L_s + y) - F(L_s))\nu(s,dy)\, ds$ is a martingale, for any bounded measurable function F.

For $F(L_t) = 1_{\{L_t \le x\}}$, we obtain $F(L_s + y) - F(L_s) = -1_{\{L_s + y > x\}} 1_{\{L_s \le x\}}$. □

(T, x)-Bonds

- Contingent claim with payoff $F(L_T)$ at T can be decomposed

$$F(L_T) = F(1) - \int_0^1 F'(y) 1_{\{L_T \leq y\}} \, dy$$

- Hence static replicating portfolio, at $t \leq T$, is

$$F(1) P(t, T) - \int_0^1 F'(y) P(t, T, y) \, dy$$

$\Rightarrow$ **(T, x)-bonds span all European type contingent claims**

Single Tranche CDOs (STCDOs)

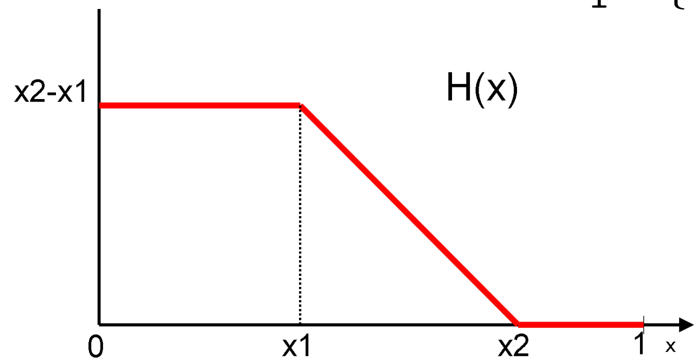
Standard instrument for investing in CDO-pool (e.g. iTraxx).

Specified by

- a number of future dates $T_0 < T_1 < \cdots < T_n$,
- a **tranche** with lower and upper detachment points $x_1 < x_2$,
- a fixed **spread** S.

Single Tranche CDOs (STCDOs)

Write $H(x) = (x_2 - x)^+ - (x_1 - x)^+ = \int_{x_1}^{x_2} 1_{\{x \le y\}} dy$



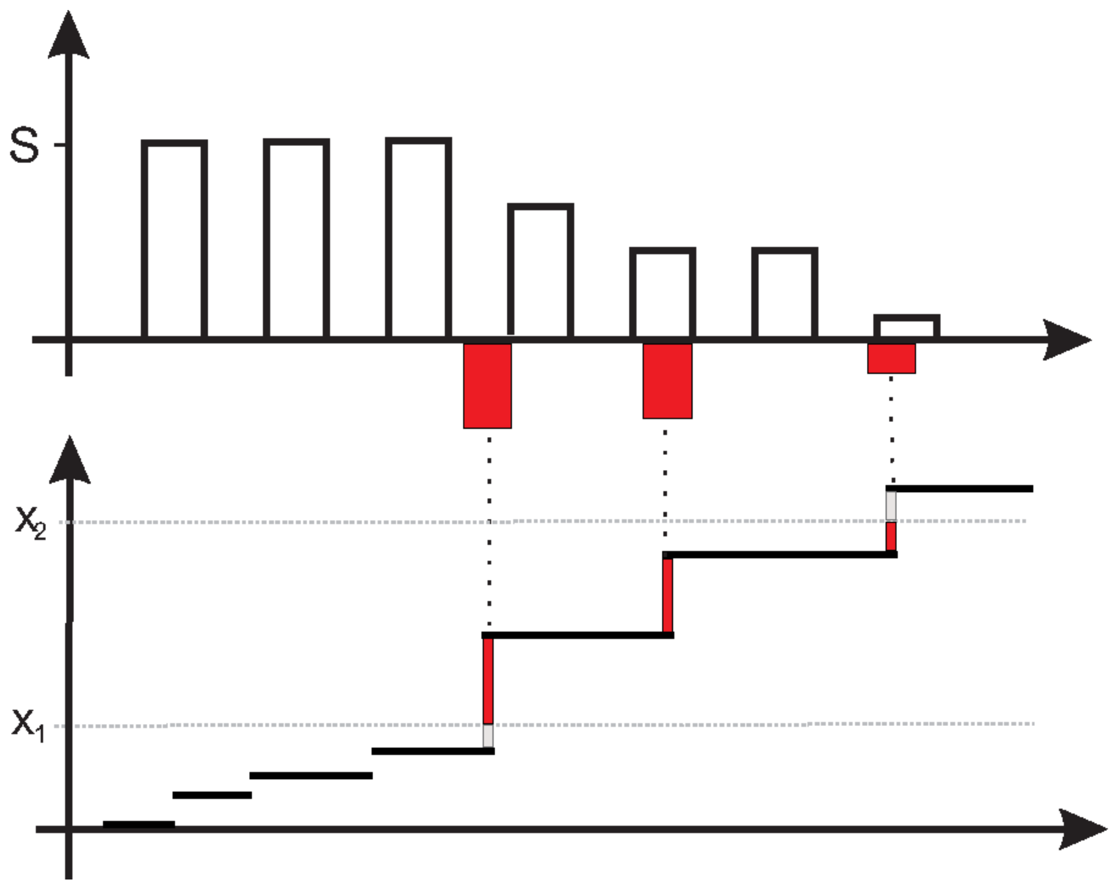
An investor in this STCDO

- **receives** $SH(L_{T_i})$ at T_i, $i = 1, \ldots, n$ (**payment leg**),
- **pays** $-\Delta H(L_t) = H(L_{t-}) - H(L_t)$ at any time $t \in (T_0, T_n]$ where $\Delta L_t \neq 0$ (**default leg**).

$\Rightarrow$ STCDO can be priced via (T, x)-bonds

Single Tranche CDOs (STCDOs)

Cash-flow attributed to tranche $(x_1, x_2]$:



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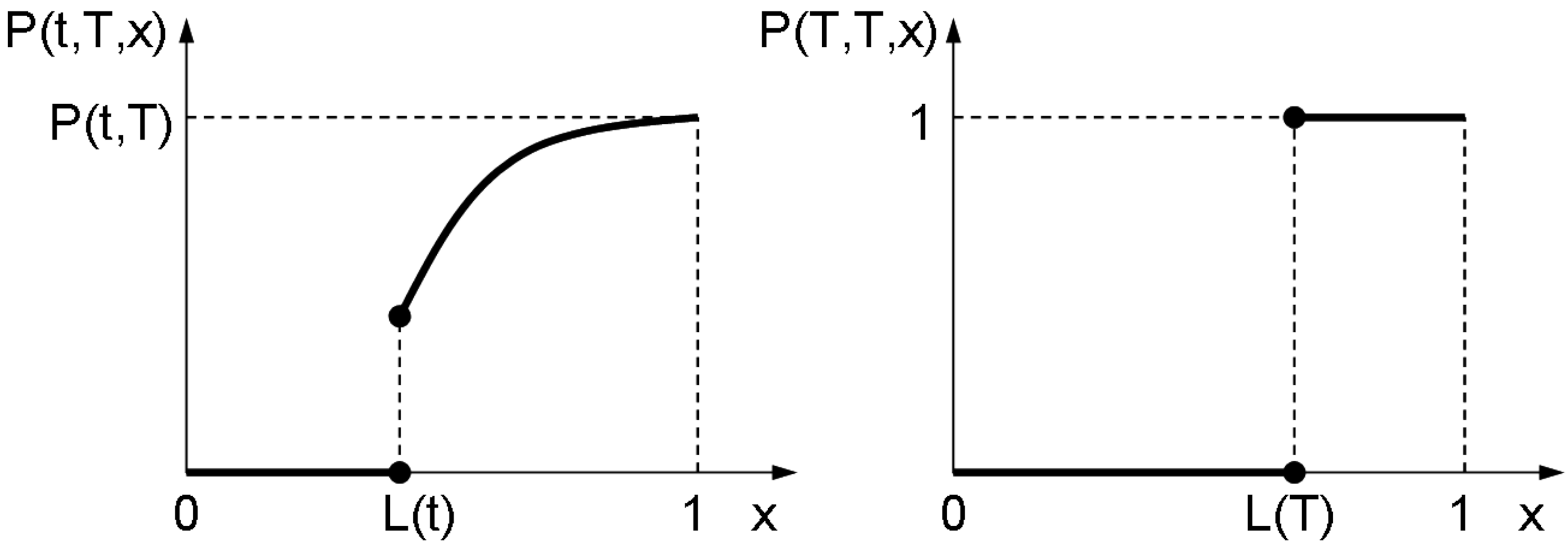
Term Structure Movements

Aim: describe (T, x)-bond price movements explicitly by

$$P(t, T, x) = 1_{\{L_t \le x\}} \mathrm{e}^{-\int_t^T f(t,u,x)\,du}$$

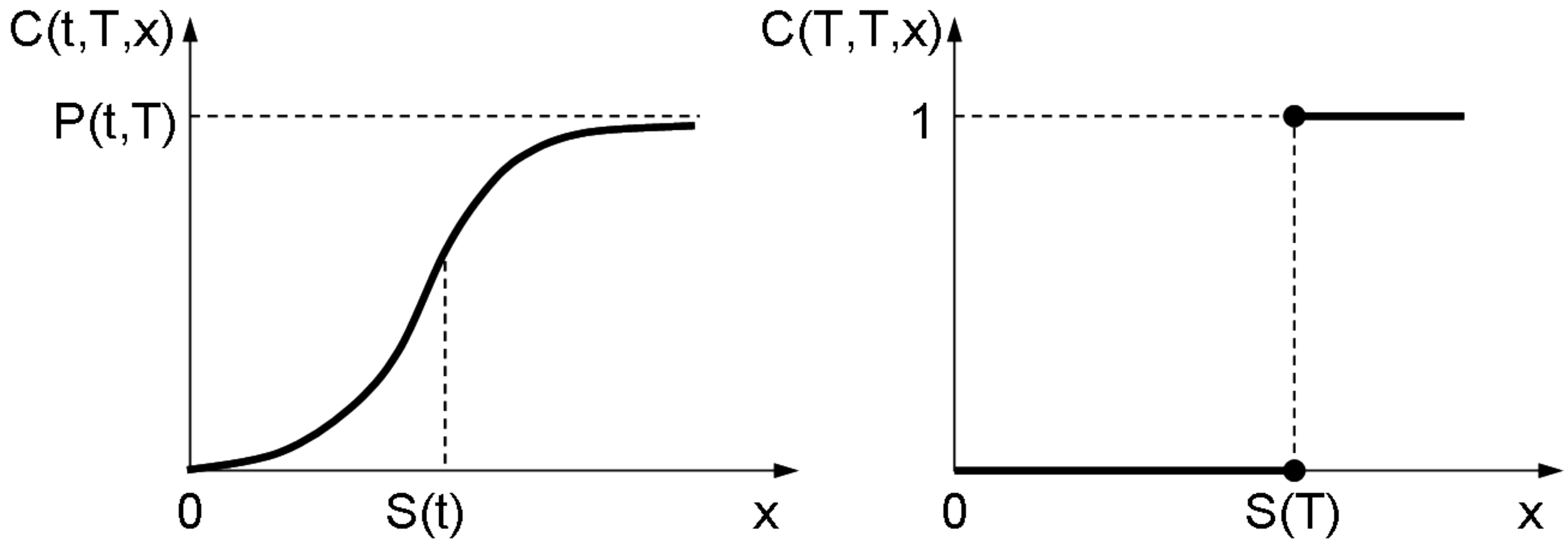


$P(t, T, x) = P(t, T)\mathbb{Q}^T [L_T \le x \mid \mathcal{F}_t]$ is $\mathcal{F}_t$-conditional CDF of L_T w.r.t. $\mathbb{Q}^T$

Note the Difference

$\mathcal{F}_t$-conditional CDF of stock price S_T w.r.t. $\mathbb{Q}^T$

$$C(t,T,x) = P(t,T)\mathbb{Q}^T[S_T \leq x \mid \mathcal{F}_t]$$



Term Structure Movements

(T,x)-bond price

$$P(t,T,x) = 1_{\{L_t \le x\}} \mathrm{e}^{-\int_t^T f(t,u,x)\,du}$$

where $f(t,T,x)$ is the **(T,x)-forward rate**

$$f(t,T,x) = f(0,T,x) + \int_0^t a(s,T,x)ds + \int_0^t b(s,T,x)^\top \cdot dW_s$$

Risk-free **T-forward rate** $f(t,T) = f(t,T,1)$

short rate $r_t = f(t,t,1)$

Term Structure Movements

Include **contagion**:

- direct: $\Delta f(t,T,x) = c(t,T,x;\Delta L_t)$
- indirect: $b(t,T,x) = b(t,T,x;L)$, same for a, c

$$f(t,T,x) = f(0,T,x) + \int_0^t a(s,T,x;\mathbf{L})ds + \int_0^t b(s,T,x;\mathbf{L})^\top \cdot dW_s$$
$$+ \sum_{\mathbf{s}\le\mathbf{t}} \mathbf{c}(\mathbf{s},\mathbf{T},\mathbf{x};\mathbf{\Delta L_s})\mathbf{1}_{\{\mathbf{\Delta L_s}>0\}}$$

Arbitrage-free Term Structure Movements

No arbitrage (NA): $\mathrm{e}^{-\int_0^t r_s\,ds} P(t,T,x)$ local martingale $\forall (T,x)$

Theorem 2: NA is equivalent to

$$\int_t^T a(t,u,x)\,du = \frac{1}{2}\left\| \int_t^T b(t,u,x)\,du \right\|^2 + \int_0^1 \left(\mathrm{e}^{-\int_t^T c(t,u,x;y)\,du} - 1 \right) \nu(t,dy),$$

$$\lambda(t,x) = f(t,t,x) - r_t$$

on $\{L_t \le x\}$, $dt \otimes d\mathbb{Q}$-a.s. for all (T,x).

NB: recall $\nu(t,dy) = -\lambda(t, L_t + dy) = -f(t,t,L_t + dy)$

Single Tranche CDOs (STCDOs)

Write $p(t,T,x) = \mathrm{e}^{-\int_t^T f(t,u,x)du}$.

Lemma 4: The value of the STCDO at time $t \le T_0$ is

$$\Gamma(t,S)$$
$$= \int_{(x_1,x_2]} 1_{\{L_t \le y\}} \left(S \sum_{i=1}^{n} p(t,T_i,y) - p(t,T_0,y) + p(t,T_n,y) + \gamma(t,y) \right) dy$$

where

$$\gamma(t,y) = \int_{T_0}^{T_n} f(t,u) p(t,u,y)\, du \quad \text{if } f(t,u) \text{ and } L_t \text{ are independent.}$$

Forward STCDO spread $S^*(t)$ defined by $\Gamma(t, S^*(t)) = 0$.

STCDO swaption with strike K has payoff at maturity T_0

$$\left(\sum_{i=1}^{n} \int_{(x_1, x_2]} 1_{\{L_t \le y\}} p(T_0, T_i, y)\, dy \right) \left(K - S^*_{T_0} \right)^+ .$$

Martingale Problem

Aim: exogenous specification of $b(t,T,x)$ and $c(t,T,x)$ determines full (T,x)-bond model $P(t,T,x)$.

Martingale problem: implicit loss process L_t such that $\nu(t,dx) = -f(t,t,L_t+dx)$ becomes compensator

Assumption: canonical stochastic basis

$\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F}_t = \mathcal{G}_t \otimes \mathcal{H}_t$, $\mathbb{Q}(d\omega_1, d\omega_2) = \mathbb{Q}_1(d\omega_1)\mathbb{Q}_2(\omega_1, d\omega_2)$:

- $(\Omega_1, \mathcal{G}, (\mathcal{G}_t), \mathbb{Q}_1)$ carrying market information, i.e. Brownian motion $W(\omega) = W(\omega_1)$,
- $(\Omega_2, \mathcal{H})$ canonical space of $[0,1]$-valued increasing MPPs, loss process = coordinate process: $L_t(\omega) = \omega_2(t)$
- $\mathbb{Q}_2$ probability kernel from Ω_1 to $\mathcal{H}$ to be determined below.

Martingale Problem

Solution: Jacod (75), "Multivariate Point Processes: Predictable Projection, Radon-Nikodym Derivatives, Representation of Martingales"

Theorem 3: Given vola and contagion parameters $b(\omega; t, T, x) = b(\omega_1, \boldsymbol{\omega_2}; t, T, x)$ and $c(\omega; t, T, x, y) = c(\omega_1, \boldsymbol{\omega_2}; t, T, x, y)$

1. Define $a(t, T, x)$ via NA drift condition.

2. Solve for $f(t, T, x)$ along **any loss path** ω_2.

3. Jacod (75): $\exists$ unique kernel $\mathbb{Q}_2$ such that NA holds.

Martingale Problem

Theorem 3 contd.: Moreover, on $\{\tau_n < \infty\}$,

$$\mathbb{Q}\left[\tau_{n+1} - \tau_n > t \mid \mathcal{G} \otimes \mathcal{H}_{\tau_n}\right] = \mathrm{e}^{-\int_{\tau_n}^{\tau_n + t} \nu(\omega_1, \omega_2(\tau_n); s, [0,1])\, ds}$$

and

$$\mathbb{Q}\left[\Delta L_{\tau_n} \in A \mid \mathcal{G} \otimes \mathcal{H}_{\tau_n -}\right] = \frac{\nu(\tau_n, A)}{\nu(\tau_n, [0,1])}, \qquad A \subset [0,1]$$

where $0 < \tau_1 < \tau_2 < \cdots$ denote jump times of L.

Monte–Carlo algorithm

Along any Brownian path $\omega_1^{(1)}, \ldots, \omega_1^{(N)}$, by recursion

- solve $f(t, T, x)$ with $L_t \equiv L_{\tau_{i-1}}$ for $t \geq \tau_{j-1}$

- set $\tau_j =$ inf $\left\{ t \mid \int_{\tau_{j-1}}^{t} \lambda(s, L_{\tau_{j-1}}) ds \geq \epsilon^{(j)} \right\}$, $\quad \epsilon^{(j)} \sim$ exp iid

- simulate $\Delta L_{\tau_j} \sim \frac{-\lambda(\tau_j, L_{\tau_{j-1}} + dx)}{\lambda(\tau_j, L_{\tau_{j-1}})}$, $\quad x \geq 0$

- restart at τ_j with $\Delta f(\tau_j, T, x) = c(\tau_j, T, x; \Delta L_{\tau_j})$

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Doubly Stochastic Framework

No contagion $b(\omega) = b(\omega_1)$ and $c = 0$.

Then L becomes (uniquely) $\mathcal{G}$-conditional Markov.

Moreover, for any $\mathcal{G}$-measurable $X \geq 0$:

$$\mathbb{E}[X \mathbb{1}_{\{L_T \leq x\}} \mid \mathcal{F}_t] = \mathbb{1}_{\{L_t \leq x\}} \mathbb{E}\left[X \mathrm{e}^{-\int_t^T \lambda(s,x) ds} \mid \mathcal{G}_t\right].$$

(This is the SPA 08 framework)

Affine Term Structure Models

State space $\mathcal{Z} \subset \mathbb{R}^d$, state process

$$dZ_t = \mu(Z_t)dt + \sigma(Z_t) \cdot dW_t,$$
$$Z_0 = z$$

Affine term structure (ATS)

$$f(t,T,x) = A'(t,T,x) + B'(t,T,x)^\top \cdot Z_t$$

Write $A(t,T,x) = \int_t^T A'(t,u,x)du$, $B(t,T,x) = \int_t^T B'(t,u,x)du$.

Affine Term Structure Models

Theorem 6: Suppose ATS and NA holds for all $z \in \mathcal{Z}$. Then ("generically") Z is **affine**:

$$\mu(z) = \mu_{\mathbf{0}} + \sum_{i=1}^{d} z_i \mu_{\mathbf{i}}, \quad \frac{1}{2}\sigma \cdot \sigma^{\top}(z) = \nu_{\mathbf{0}} + \sum_{i=1}^{d} z_i \nu_{\mathbf{i}}$$

and A and B solve **Riccati equations**, for $t \leq T$,

$$-\partial_t A(t,T,x) = \mathbf{A}'(\mathbf{t},\mathbf{t},\mathbf{x}) + {\mu_{\mathbf{0}}}^{\top} \cdot B(t,T,x) - B(t,T,x)^{\top} \cdot \nu_{\mathbf{0}} \cdot B(t,T,x)$$
$$-\partial_t B_i(t,T,x) = \mathbf{B}'_{\mathbf{i}}(\mathbf{t},\mathbf{t},\mathbf{x}) + {\mu_{\mathbf{i}}}^{\top} \cdot B(t,T,x) - B(t,T,x)^{\top} \cdot \nu_{\mathbf{i}} \cdot B(t,T,x)$$

with $A(T,T,x) = 0$ and $B(T,T,x) = 0 \ \forall (T,x)$.

Affine Term Structure Models

Theorem 7: Conversely, suppose Z is affine, and let $\mathbf{A'(t,t,x)}$, $\mathbf{B'(t,t,x)}$ be bounded functions such that $\mathbf{A'(t,t,x)}+\mathbf{B'(t,t,x)}^\top \cdot z$ is decreasing and càdlàg in x for all t and $z \in \mathcal{Z}$.

Let A and B be given as solutions of the **Riccati equations**. Then

$$P(t,T,x) = 1_{\{L_t \leq x\}} \mathrm{e}^{-A(t,T,x)-B(t,T,x)^\top \cdot Z_t}$$

defines an arbitrage-free (T,x)-bond market.

Affine Term Structure Models

Simple example: $dZ_t = (\mu_\mathbf{0} + \mu_\mathbf{1} Z_t)dt + \sigma\sqrt{Z_t}dW_t$.

Moreover: $\mathbf{A'(t,t,x)} = \alpha\mathbf{(t,x)}$ with $\alpha\mathbf{(t,1)} \equiv \mathbf{r} \geq 0$, $\mathbf{B'(t,t,x)} = \beta\mathbf{(x)}$ with $\beta\mathbf{(1)} \equiv \mathbf{0}$, so that:

$$r_t \equiv \mathbf{r}, \quad \text{and} \quad \lambda(t,x) = \alpha\mathbf{(t,x)} - \mathbf{r} + \beta\mathbf{(x)} Z_t.$$

The **Riccati equations** become

$$A(t,T,x) = \int_t^T \left(\alpha\mathbf{(s,x)} + \mu_\mathbf{0} B(s,T,x)\right) ds$$

$$-\partial_t B(t,T,x) = \beta\mathbf{(x)} + \mu_\mathbf{1} B(t,T,x) - \frac{\boldsymbol{\sigma}^\mathbf{2}}{\mathbf{2}} B(t,T,x)^2, \quad B(T,T,x) = 0$$

with solution

$$B(t,T,x) \equiv B(T-t,x) = \frac{2\beta(\mathbf{x})\left(e^{\rho(\mathbf{x})(T-t)} - 1\right)}{\rho(\mathbf{x})\left(e^{\rho(\mathbf{x})(T-t)} + 1\right) - \mu_{\mathbf{1}}\left(e^{\rho(\mathbf{x})(T-t)} - 1\right)}$$

where $\rho(\mathbf{x}) = \sqrt{\mu_{\mathbf{1}}^{\mathbf{2}} + 2\sigma^{\mathbf{2}}\beta(\mathbf{x})}$.

We obtain

$$f(t,T) \equiv \mathbf{r}$$
$$f(t,T,x) = \alpha(\mathbf{T},\mathbf{x}) + \mu_{\mathbf{0}} B(T-t,x) + \partial_T B(T-t,x) Z_t.$$

$\longrightarrow$ efficient computation of STCDO values and swaptions

$\longrightarrow$ matches any initial spread curve $f(0,T,x)$ by choice of $\alpha(\mathbf{T},\mathbf{x})$.