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COMBINED METHODOLOGY FOR MODELLING AND MEASURING OPERATIONAL RISK

1. Fitting severity distributions
   - Using EVT basic concepts
   - Bayesian methodology as a proper way for mixing internal and external data
2. Modelling loss frequency
   - Poisson and compound Poisson models
   - Modifications of "short memory process"
3. Evaluation of aggregate loss distribution
   - FFT algorithm as an efficient way for aggregate loss calculation
   - Comparison of different methods

Operational risk

*The risk of losses resulting from inadequate or failed internal processes, people and systems or from external events *.*

- Basic Indicator Approach (BIA)
- Traditional Standardised Approach (TSA)
- Advanced Measurement Approaches (AMA).

Business Line 1	Corporate finance
Business Line 2	Trading and sales
Business Line 3	Retail banking
Business Line 4	Commercial banking
Business Line 5	Clearing
Business Line 6	Agency services
Business Line 7	Asset Management
Business Line 8	Retail brokerage
Business Line 9	Private banking
Business Line 10	Corporate items

*[Basel Committee on Banking Supervision (2001).
Consultative Document. Overview of the New Basel Capital Accord.]

The Problem

GIVEN	TO BE FOUND
• Database of historical **internal** OpLoss data from selected bank • **External** OpLoss data from "ORX" bank association • All the data classified w.r.t. *Lines of Business* (BL) and *Types of Events* (ET)	α–VaR for yearly aggregate losses $\mathbf{VaR}_\alpha(S) = \inf\{s \in \mathbb{R} : \mathbf{P}(S > s) \leq 1-\alpha\}$ • Capital charge for OpRisk must be calculated as $C = \sum_{l=0}^{K} VaR_\alpha^l,$ where $l = 1, \ldots, K$ denotes BL's.

METHODS

- Severity distributions for BL's are very heavy tailed $\rightarrow$ EVT is used
- Homogeneous and inhomogeneous Poisson processes,
  Negative Binomial distribution are used to model loss frequency
- To evaluate aggregate loss several methods are used:
  Monte–Carlo modelling, FFT approach, Recursion formulae

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Fitting severity distributions

Generalized Pareto distribution (GPD)

$$G_{\xi,\mu,\beta}(x) = 1 - \left(1 + \xi\frac{x-\mu}{\beta}\right)^{-1/\xi} \tag{1}$$

μ location parameter (threshold), β scale parameter, ξ shape parameter.

The density for GPD

$$g_{\xi,\mu,\beta}(x) = \frac{1}{\beta}\left(1 + \xi\frac{x-\mu}{\beta}\right)^{-\frac{1}{\xi}-1} \tag{2}$$

Log–likelihood function for MLE parameter estimation

$$l_g(\theta) = L_g((\xi,\beta);\mathbf{X}) = -n\ln\beta - \left(\frac{1}{\xi}+1\right)\sum_{j=1}^{n}\ln\left(1+\frac{\xi}{\beta}(X_j-\mu)\right)\ ; \tag{3}$$

Pareto distribution

$$P_{\xi,\mu,\beta}(x) = 1 - \left(1 + \frac{x-\mu}{\beta}\right)^{-1/\xi} \tag{4}$$

Three–parameter **Weibull distribution**

$$W_{\xi,\mu,\beta}(x) = 1 - e^{\left(\frac{-(x-\mu)}{\beta}\right)^{\xi}}, \tag{5}$$

$$\text{density: } w_{\xi,\mu,\beta}(x) = \xi\,\beta^{-\xi}\,x^{\xi-1}\,e^{[-(x-\mu)/\beta]^{\xi}}. \tag{6}$$

$$\text{Log–likelihood: } l_w(\theta) = \ln\xi - \xi\ln\beta + \sum_{i=1}^{n}\left((\xi-1)\ln X_i - \left[\frac{X_i-\mu}{\beta}\right]^{\xi}\right) \tag{7}$$

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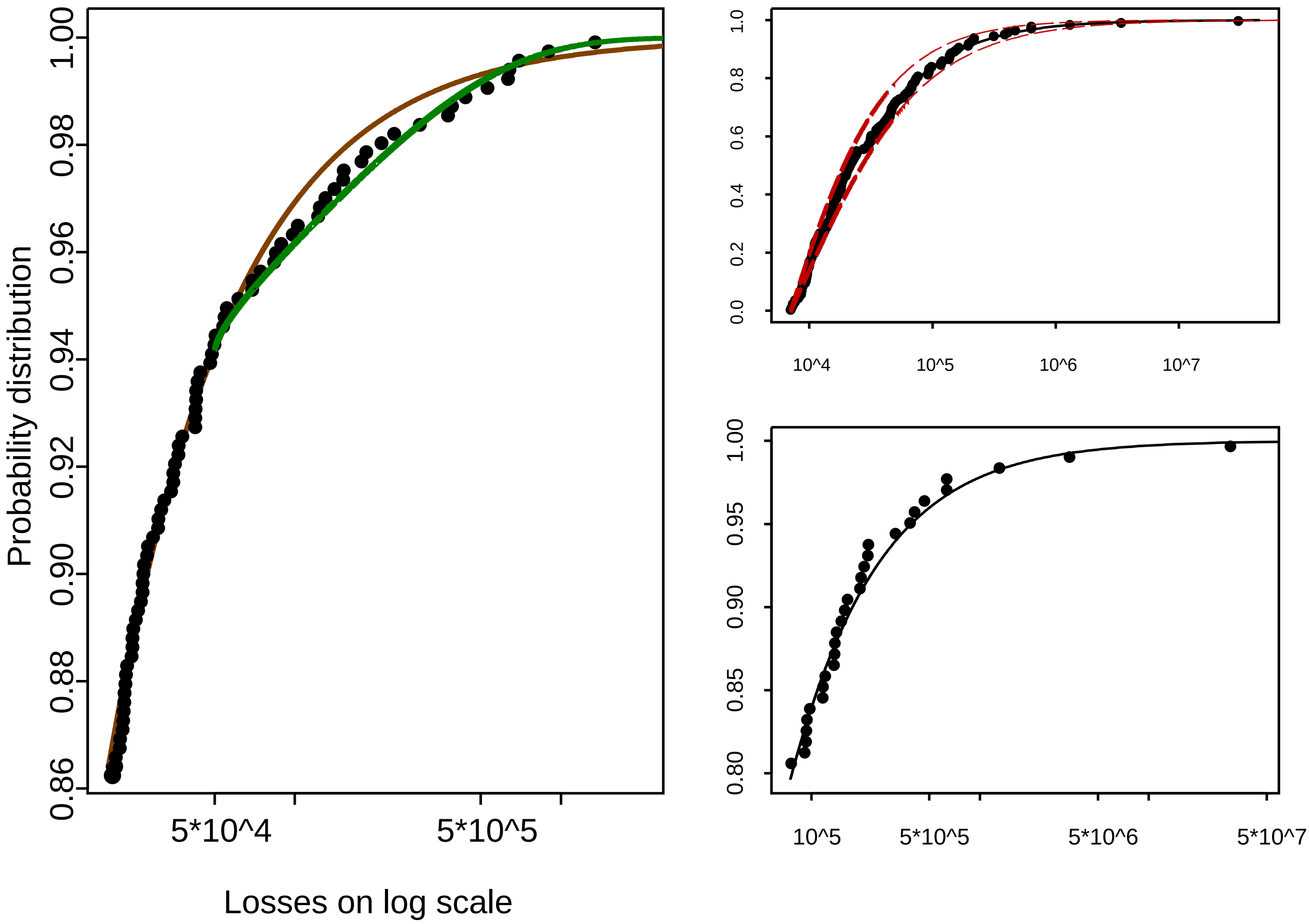
Figure 1. Fit with analytical distributions for selected BL's,

Top right: BL 10 — GPD fit with error bounds,
bottom right — tail of BL 10 fit,
Left: BL2 — Combination of GPD (brown)
and Weibull (green) — for the tail



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Mixing internal and external data

Bayesian inference

$$\pi_{\theta\,|\,X}(\theta\,|\,\mathbf{x}) = \frac{f_{X\,|\,\theta}(\mathbf{x}\,|\,\theta)\pi(\theta)}{\int f_{X\,|\,\theta}(\mathbf{x}\,|\,\theta)\pi(\theta)d\theta}, \tag{8}$$

$\pi_{\theta\,|\,X}(\theta\,|\,\mathbf{x})$ — posterior parameter distribution

$\pi(\theta)$ — prior parameter distribution

For the case of **Pareto** and **Poisson** joint model

- Prior for Poisson intensity λ — gamma distribution $\gamma_{a,b}(\lambda)$
- Prior for shape parameter ξ — gamma distribution $\gamma_{c,d}(\xi)$
- Prior for scale parameter β — reciprocal gamma $p(\beta) = \mathbf{P}(B = \beta)$ ($\mathbf{P}(1/B = \beta) = \gamma_{e,f}(\beta)$)

Then Posterior density for scale parameter β:

$$\tilde{p}(\beta) = \beta^{-k}(d'(\beta))^{-(s+k)} \exp\left(-\sum_{i\le k}\log(1 + X_i/\beta)\right) p(\beta),$$

where
$a' = a + k$, $b' = b + T$, $c' = c + k$
and $d'(\beta) = d + \sum_{i\le k}\log(1 + X_i/\beta)$

[R.-D.Reiss, M.Thomas, 1999]

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Mixing internal and external data

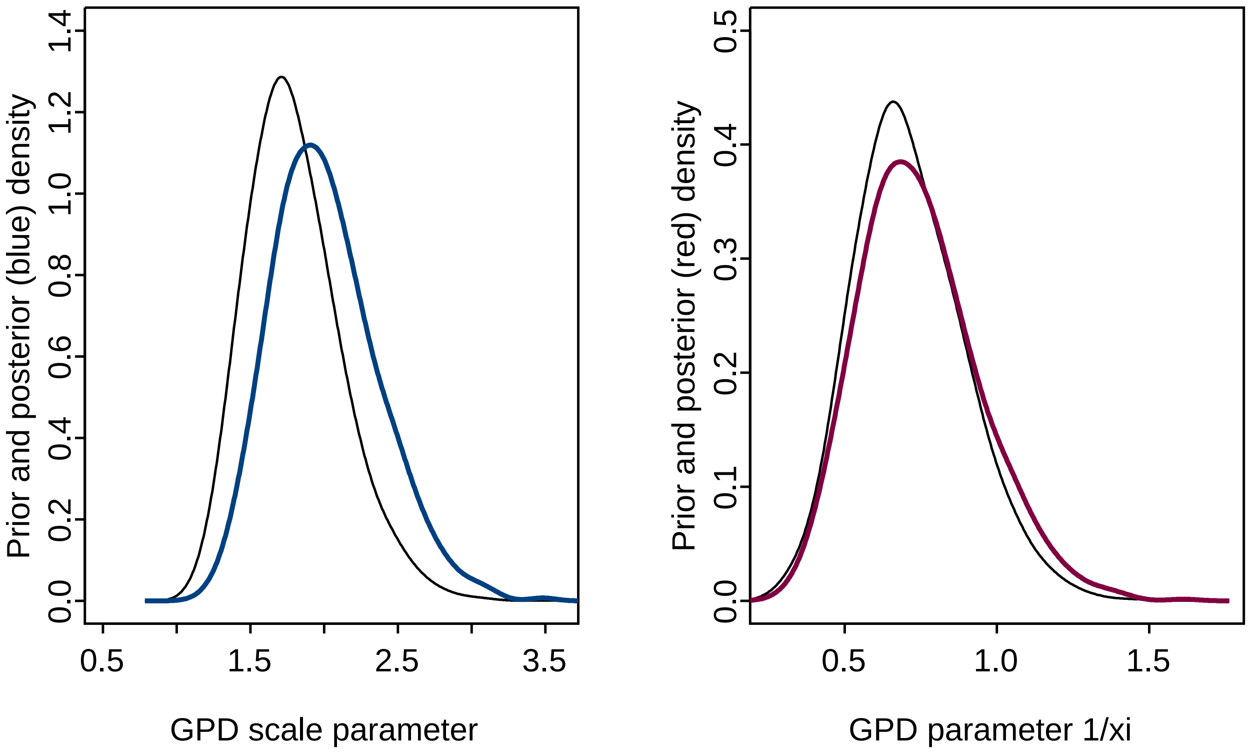
Bayesian estimators for Pareto parameters ξ and β and Poisson intensity λ

$$\lambda^* = \frac{a+k}{b+T} \tag{9}$$

$$\xi^* = \int \frac{c'}{d'(\beta)} \tilde{p}(\beta) d\beta \tag{10}$$

$$\beta^* = \int \beta \tilde{p}(\beta) d\beta \tag{11}$$

Figure 2. Bayesian parameter estimators, BL 7. Posterior versus prior



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Modelling loss frequency (point processes)

Point process of exceedances:

$\{(T_i, X_i),\, 1 \le i \le N\,,\, X_i > u\}$ in $(0,T) \times (u,\infty)$

Possible models for T_i in OpRisk

- homogeneous Poisson process $\mathbf{P}(N(t)=k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$, $\mathbf{P}(T_i < x) = 1 - \exp(-\lambda x)$
- negative binomial $\mathbf{P}(N(t)=k) = \begin{pmatrix} k+r-1 \\ k \end{pmatrix} \left(\frac{1}{1+\beta}\right)^r \left(\frac{\beta}{1+\beta}\right)^k$
- inhomogeneous Poisson process $\lambda = \lambda(t)$
- jump process with short memory $\lambda_t = \lambda_0 + \rho\lambda_{t-1} + \eta\zeta(t)$, $\zeta(t) = \mathbf{E}[N_{t-1} \,|\, I_{t-1}] - \lambda_{t-1}$
- Dirichlet process $T(t) - T(s) = \Gamma\left(\alpha(t) - \alpha(s), \beta\right)$

Checking POT model assumptions

with GPD for severity and Poisson for frequency

$$W_i = \frac{1}{\xi}\log\left(1 - \frac{X_i^u - \mu}{\beta - \xi(u-\mu)}\right), \quad Z_k = \lambda \times (T_k - T_{k-1}), \quad i,k = 1,\ldots,N \tag{11}$$

Intensity for point process of exceedances

$$\lambda_u = \lambda_0 \mathbf{P}(X_i > u) \tag{12}$$

$$\lambda_u = \lambda_u(t) \overset{?}{\Longleftrightarrow} \xi = \xi(t),\ \beta = \beta(t),\ \text{or } \lambda_0 = \lambda_0(t)$$

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Diagnostics for selected model

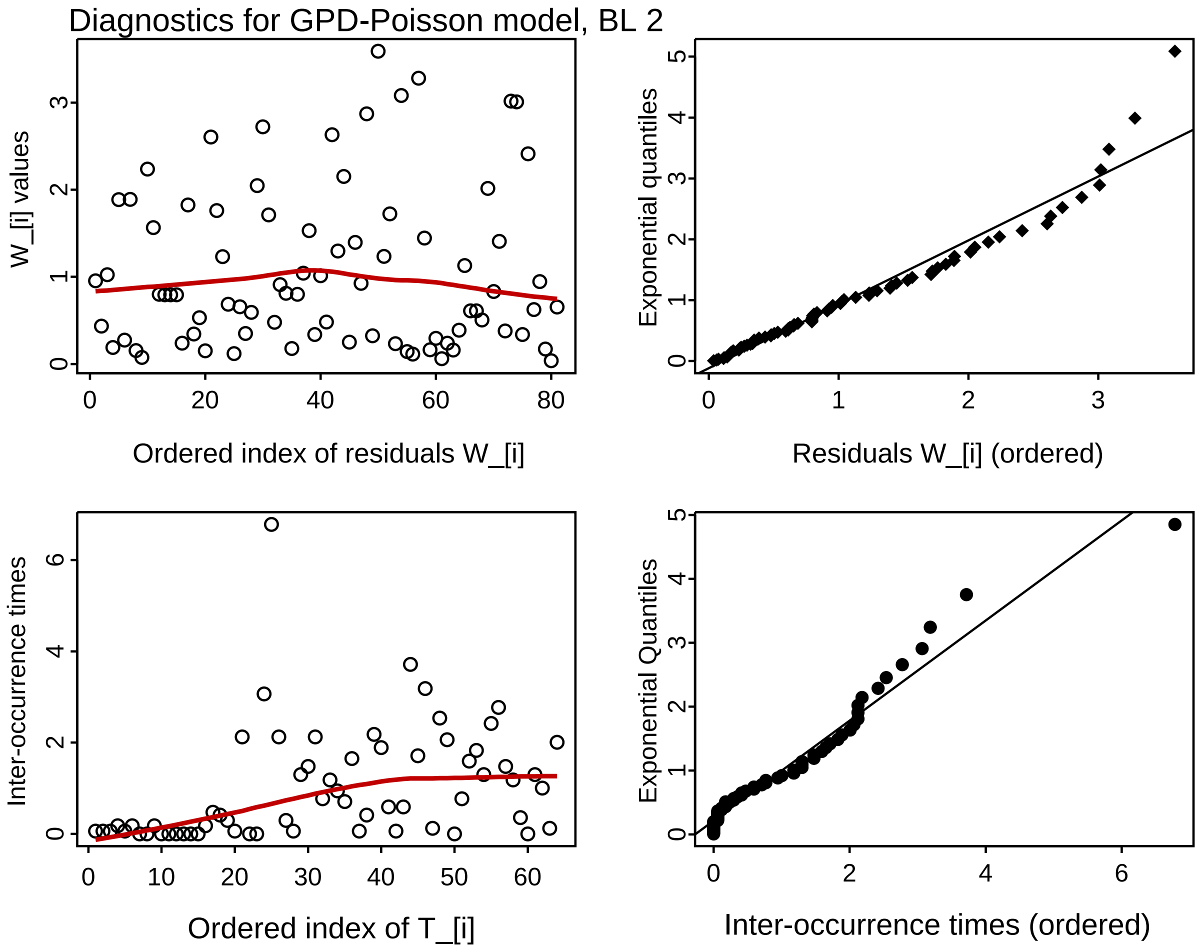


Figure 3. Checking GPD–Poisson model assumptions

Methodology proposed by [AJ McNeil, T. Saladin (2000)]

MLE parameter estimation for loss counting processes

$$\text{Poisson}: l = \sum_{k=0}^{\infty} n_k \ln p_k = \sum_{k=0}^{\infty} n_k \ln\left(\frac{e^{-\lambda}\lambda^k}{k!}\right) = \sum_{k=0}^{\infty} n_k(-\lambda + k\ln\lambda - \ln(k!)), \tag{13}$$

$$\text{NegBin}: l = \sum_{k=0}^{\infty} n_k \left[\ln\binom{r+k-1}{k} - r\,\ln(1+\beta) + k\,\ln\beta + k\,\ln(1+\beta)\right], \tag{14}$$

- k — number of events during one year
- n_k — number of years in which exactly k losses occurred.

MLE results BL 2 $u = 7000$	Hom. Poisson	Neg. Binomial	InHom.Poisson linear intensity	Short memory process
Parameters estimated	$\lambda = 28.3$	$r = 19.12$ $\beta = 0.34$	$\lambda = a + b \cdot t$ $a = 29,\ b = -0.88$	$\lambda_0 = 0.04,\ \rho = 0.94,$ $\eta = 0.36$
Loglikelihood	-55.57	-54.86	$\mathbf{-50.21}$	-52.64
AIC	113.14	113.72	**104.42**	111.28

Jump process with negative binomial distribution and bayesian inference

Given random sample of size N, NB estimators $\begin{pmatrix} r \\ \beta \end{pmatrix}$

For each of N occurrences the following can be defined:

Bayesian estimator 1) w.r.to occurrence n $\begin{pmatrix} \widehat{r}_n \\ \widehat{\beta}_n \end{pmatrix}$ 2) w.r.to occurrence $1 \ldots n$ $\begin{pmatrix} \widehat{r}_{0n} \\ \widehat{\beta}_{0n} \end{pmatrix}$

$$\begin{pmatrix} \widetilde{r}_{n+1} \\ \widetilde{\beta}_{n+1} \end{pmatrix} = a + b\begin{pmatrix} \widehat{r}_n \\ \widehat{\beta}_n \end{pmatrix} + c\begin{pmatrix} \widehat{r}_{0n} \\ \widehat{\beta}_{0n} \end{pmatrix} \tag{12}$$

MLE results BL 2 $u = 7000$	NB–Bayes
Parameters estimated	$a = 0.07,\ b = 0.43,$ $c = 0.46$
Loglikelihood	$\mathbf{-49.4}$
AIC	**104.8**

Calculating aggregate loss distribution $S = X_1 + \cdots + X_{N(t)}$

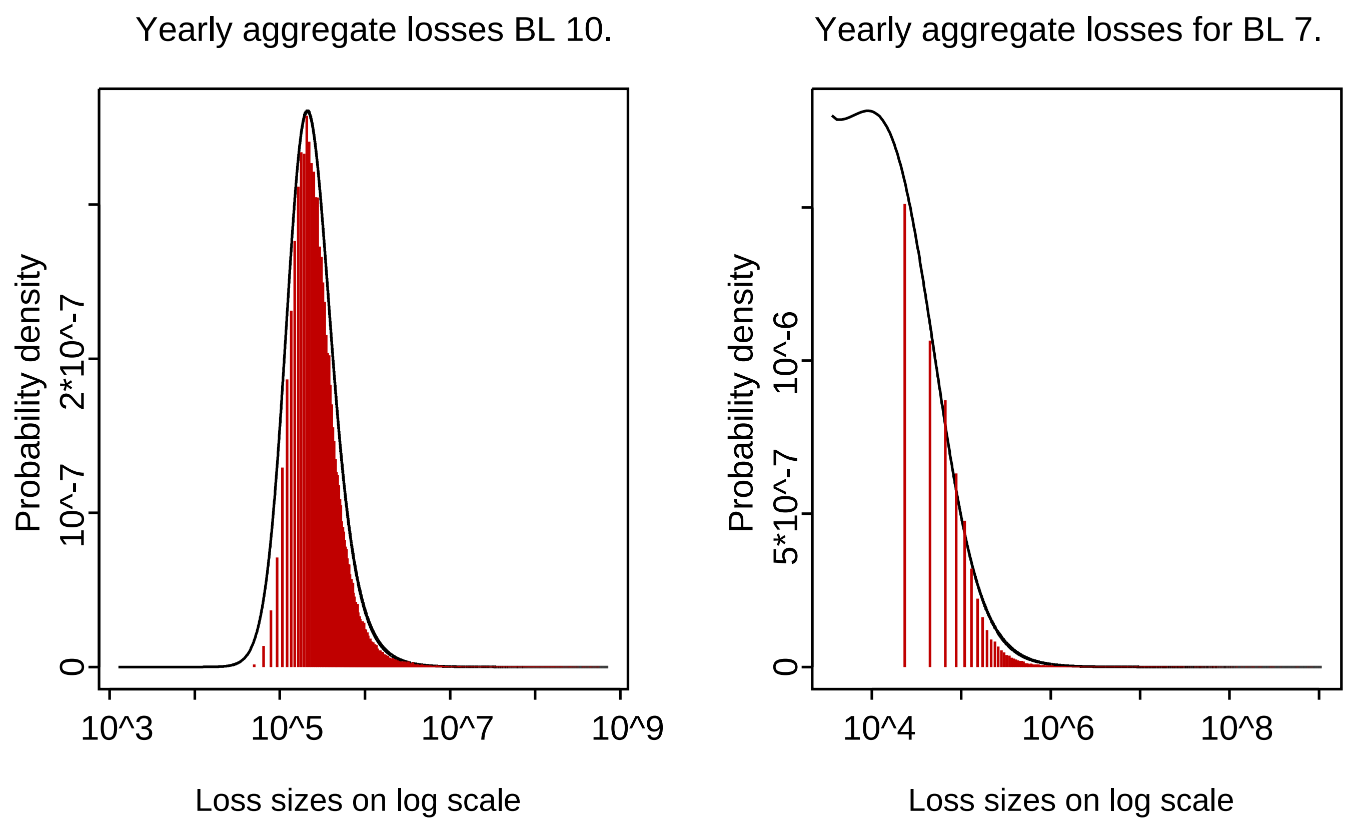
Characteristic function for yearly aggregate losses

$$\widehat{g}_S(u) = \exp\left\{-\lambda t(1 - \widehat{f}_X(u))\right\}, \quad \widehat{f}_X(u) = \int_{-\infty}^{\infty} e^{iux} dF_X(x) \tag{15}$$

Density calculation via **FFT application**

$$f_k \equiv f_X(x_k)\,,\ x_k = \mu + (q_M - \mu)k/K\,,\ g_k = \mathrm{IFFT}\left[\ \exp\left\{-\lambda t(1 - \mathrm{FFT}\,[\,x_k\,])\right\}\right]$$

Figure 4. Density for aggregate losses BL 10 ($\lambda = 28.3$), BL 7 ($\lambda = 4.7$). Red histogram-type — Monte-Carlo results, solid line — FFT results



Comparison with results obtained with *CreditRisk*$^+$ algorithm (modified Panjer recursion)

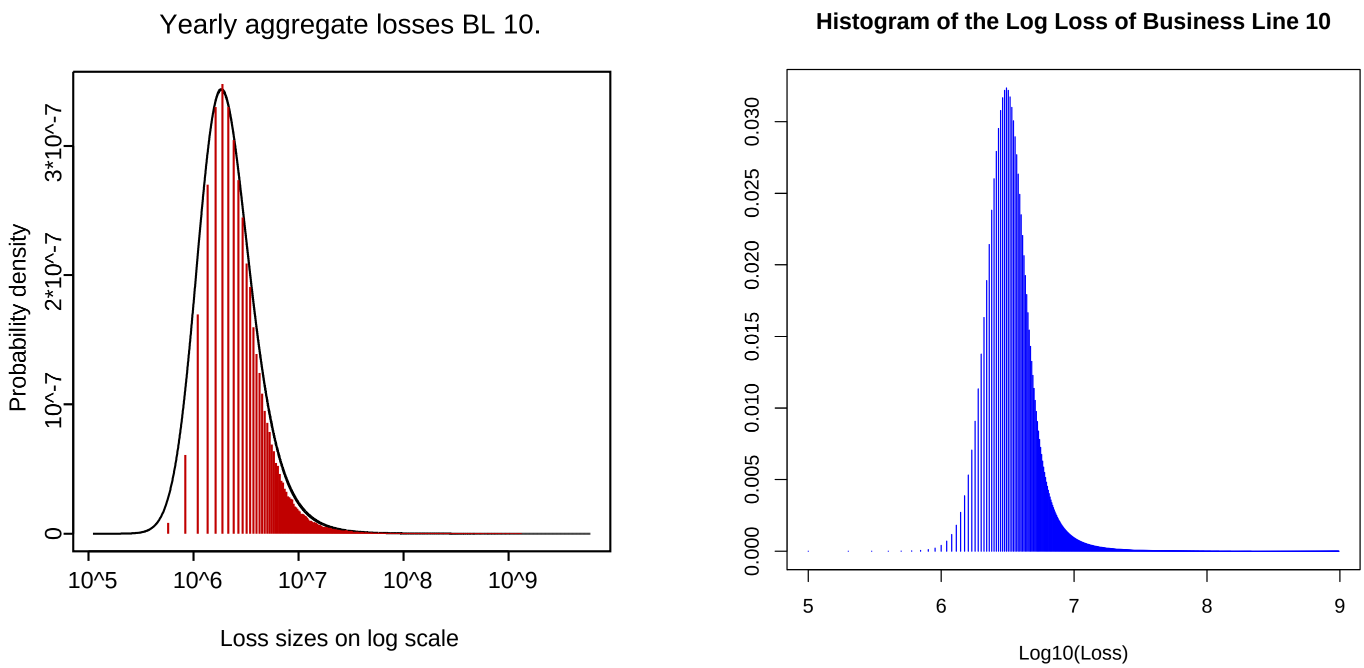


Figure 3. BL 10. Left: density plots, red histogram-type — Monte-Carlo results, solid line — FFT results. Right: Histogram, obtained with CreditRisk$^+$.

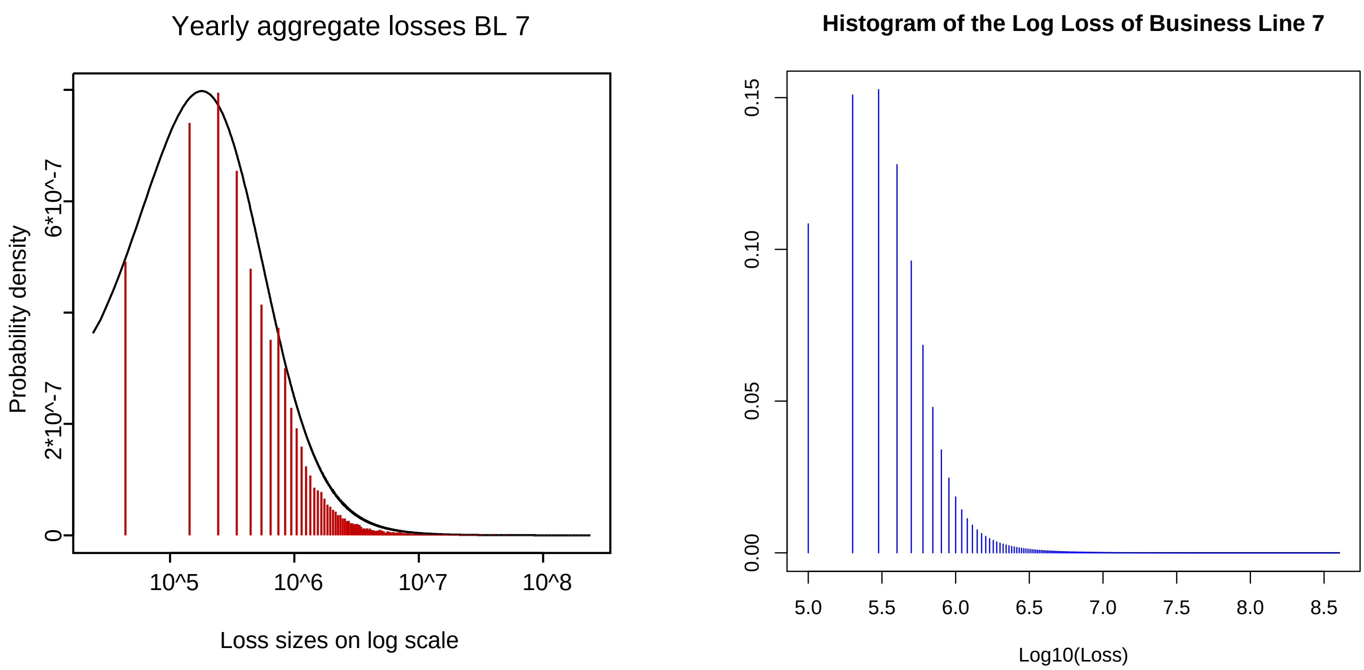


Figure 4. BL 7. Density plots in the same order as on the previous figure.

[U. Schmock. (2006). Modelling Dependent Credit Risks with Extensions of CreditRisk$^+$ and Applications to Operational Risk (Lecture Notes)]

Error bounds estimation

$$
\begin{aligned}
I(\overline{\theta})_{rs} = -\mathbf{E}H(\overline{\theta}) = -\mathbf{E}\left[\frac{\partial^2}{\partial\theta_s\partial\theta_r}l(\overline{\theta})\right] = \mathbf{E}\left[\frac{\partial}{\partial\theta_s}l(\overline{\theta})\frac{\partial}{\partial\theta_r}l(\overline{\theta})\right] \\
= \quad -n\mathbf{E}\left[\frac{\partial^2}{\partial\theta_s\partial\theta_r}\ln f(X;\overline{\theta})\right] = n\mathbf{E}\left[\frac{\partial}{\partial\theta_s}\ln f(X;\overline{\theta})\frac{\partial}{\partial\theta_r}\ln f(X;\overline{\theta})\right] \qquad (13)
\end{aligned}
$$

($H(\overline{\theta})$ — *Hessian matrix*, $I(\overline{\theta})_{rs}$ — *Fisher information matrix*)

Cramér–Rao lower bound :

$$
\mathrm{Var}(\widehat{\overline{\theta}}) \geq \left(-\mathbf{E}\left[\frac{\partial^2}{\partial\theta_s\partial\theta_r}l(\overline{\theta})\right]\right)^{-1} \qquad (14)
$$

95%–**confidence interval :**

$$
1-\alpha = \mathbf{P}\left(\widehat{\overline{\theta}} - 1.96\sqrt{v(\widehat{\overline{\theta}})} < \theta < \widehat{\overline{\theta}} + 1.96\sqrt{v(\widehat{\overline{\theta}})}\right), \qquad (15)
$$

Line No	**VaR** at 0.999 (via FFT)	lower bound	upper bound
Line 1	29	17	51
Line 2	116	30.5	159
Lines 3,8,9	139	69.5	193
Line 4	58	26.5	319
Line 6	144.5	51.3	576
Line 7	293	121	707.5
Line 10	651.5	119	**4316**
Line 11	20.5	11	48.5

Insurance problem:
selecting particular cells of "BL-ET" to be insured

Total number of loss event during 2002—2004

	ET1	ET2	ET3	ET4	ET5	ET6	ET7
BL 2						12	**51**
BLs 3,8,9	11	30		10	3	26	30
BL 4	5	7		5			**10**
BL 6						**3**	4
BL 7				6			4
BL 10	4	6	**8**	3	2		16

BL2, ET7 : $\mathbf{VaR}_{0.999} = 129\ [34, 291]$, $\mathbf{VaR}_{0.75} = 1.2\ [0.76, 2.1]$,

BL4, ET7 : $\mathbf{VaR}_{0.999} = 94\ [22, 213]$, $\mathbf{VaR}_{0.75} = 0.79\ [0.35, 1.8]$,

BL6, ET6 : $\mathbf{VaR}_{0.999} = 115\ [14, 570]$, $\mathbf{VaR}_{0.75} = 0.87\ [0.22, 2.8]$,

BL10, ET3: $\mathbf{VaR}_{0.999} = 141\ [35, 510]$, $\mathbf{VaR}_{0.75} = 1.45\ [0.77, 2.74]$,

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