Finite State Space Representation of Forward Interest Rates on a Foreign Exchange Market

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Motivation...

- Global markets development brings the need for creating tractable pricing models on international integrated market.
- We would like the model to be **flexible** enough to capture realistic features of the international bond market
- We would like the model to be **tractable** enough to produce closed form solutions and allow for a "good" fit to observed yields.

Basic definitions

- $p(t, x)$ and $\tilde{p}(t, x)$ - time t prices of the domestic and foreign zero coupon bond maturing at $T = t + x$

- $r(t, x)$ - domestic forward interest rate contracted at time t with maturity $t + x$

$$r(t, x) = -\frac{\partial \log p(t, x)}{\partial x}$$

- $\tilde{r}(t, x)$ - foreign forward interst rate contracted at time t with maturity $t + x$

$$\tilde{r}(t, x) = -\frac{\partial \log \tilde{p}(t, x)}{\partial x}$$

The international market

- We model the dynamics of domestic and foreign forward rates by (*i=domestic, foreign)*

$$df^i(t,T) = \alpha^i(t,T)dt + \sigma^i(t,T)dW_t^i$$

- We use the Musiela parameterization and HJM drift condition to rewrite:

❖ The domestic forward rate dynamics under the domestic risk neutral measure

$$dr_t(x) \quad = \quad \left\{ \frac{\partial}{\partial x} r_t(x) + \sigma_t(x) \int_0^x \sigma_t^\star(u)du \right\} dt + \sigma_t(x)dW_t.$$

❖ The foreign forward rate dynamics under the domestic risk neutral measure

$$d\tilde{r}_t(x) \quad = \quad \left\{ \frac{\partial}{\partial x} \tilde{r}_t(x) + \tilde{\sigma}_t(x) \int_0^x \tilde{\sigma}_t^\star(u)du - \tilde{\sigma}_t(x)\delta_t^\star \right\} dt + \tilde{\sigma}_t(x)dW_t,$$

Exchange rate dynamics

- Case 1: We model **spot** exchange rate Q^d - dynamics

$$dS_t = S_t(r_t(0) - \tilde{r}_t(0))dt + S_t\delta_t dW_t.$$

- Case 2: We model **forward** exchange rate dynamics under the domestic risk neutral measure

Forward exchange rate, *G(t,T),* contracted at time t, is defined as units of domestic currency that will be paid per unit of foreign currency at time T.

$$dG(t,T) = \alpha_G(t,T)dt + \delta(t,T)dW_t$$

System with spot exchange rate

$$
\begin{aligned}
dr_t &= (\mathbf{F}r_t + \sigma(\hat{r}_t)\mathbf{H}\sigma^\star(\hat{r}_t))dt + \sigma(\hat{r}_t)dW_t, \\
d\tilde{r}_t &= (\mathbf{F}\tilde{r}_t + \tilde{\sigma}(\hat{r}_t)\mathbf{H}\tilde{\sigma}^\star(\hat{r}_t) - \tilde{\sigma}(\hat{r}_t)\delta^\star(\hat{r}_t))dt \\
&+ \tilde{\sigma}(\hat{r}_t)dW_t, \\
dY_t &= (\mathbf{B}r_t - \mathbf{B}\tilde{r}_t)dt + \delta(\hat{r}_t)dW_t,
\end{aligned}
$$

where

$$\mathbf{F} = \frac{\partial}{\partial x}, \quad \mathbf{B}f(x) = f(0)$$

$$\mathbf{H}f(x) = \int_0^x f(s)ds$$

Forward exchange rate dynamics

- Forward exchange rate can be expressed as

$$G(t,T) = \mathbb{E}^{T^d}\left[S(T)\middle|\mathcal{F}_t\right]$$

where expectation is taken under risk neutral measure Q^T for numeraire process *p(t,T))*

- Under Q^T *G(t,T)* is a martingale, thus the dynamics gives us the drift condition

$$dG(t,T) = \left\{\alpha_G(t,T) - \delta(t,T)\int_t^T \sigma^\star(t,s)ds\right\}dt + \delta(t,T)dW_t^T$$

$$\alpha_G(t,T) = \delta(t,T)\int_t^T \sigma^\star(t,s)ds$$

System with forward exchange rate

$$\begin{cases} dr_t &= \{\mathbf{F}r_t + \sigma_t \mathbf{H}\sigma_t^\star\}\,dt + \sigma_t dW_t \\ d\tilde{r}_t &= (\{\mathbf{F}\tilde{r}_t + \tilde{\sigma}_t \mathbf{H}\tilde{\sigma}_t^\star - \tilde{\sigma}_t \mathbf{B}\delta_t^\star\}\,dt + \tilde{\sigma}_t dW_t \\ dG_t &= \{\mathbf{F}G_t + \delta_t \mathbf{H}\sigma_t^\star\}\,dt + \delta_t dW_t \\ r_0 &= r_0^0, \quad \tilde{r}_0 = \tilde{r}_0^0, \quad G_0 = g_0^0. \end{cases}$$

where

$$\mathbf{F} = \frac{\partial}{\partial x}, \quad \mathbf{B}f(x) = f(0)$$

$$\mathbf{H}f(x) = \int_0^x f(s)ds$$

$$\delta(x) = \int_0^x \sigma(t,s)ds - \int_0^x \tilde{\sigma}(t,s)ds + \delta_S(t)$$

Parameterized families of forward rate curves

- It became very popular to use parameterized families of smooth forward rate curves when it comes to fitting forward rate curves to initial data

- Bank of Finland and Italy are using Nelson-Siegel family

$$G_{NS}(x,z) = z_1 + (z_2 + z_3 x)\exp\{-z_4 x\}$$

- Canada, Germany, France, UK use Svensson (Nelson-Siegel extended) family

$$G_S(x,z) = z_1 + (z_2 + z_3 x)e^{-z_4 x} + z_5 x e^{-z_6 x}$$

Problems...

- Assume that we specified some interest rate model $\mathcal{M}$ and parameterized family of forward rate curves $\mathcal{G}$

- The pair $(\mathcal{M}, \mathcal{G})$ is **consistent** if all forward curves which may be produced by the interest rate model $\mathcal{M}$ are contained within the family, that is, $\mathcal{M}$ is an invariant manifold under the action of forward rates driven by

- If the pair is inconsistent, then the model $\mathcal{M}$ will produce forward curves outside the family used in the calibration step 0.

- Thus, parameters has to be changed not only because the interest rate model is an **approximation** of reality, but because the family "does not go well" with the model.

Finite Dimensional Realization Problem

- The problem of existence of an invariant manifold turns out to be equivalent to the problem of determining whether a forward rate process possesses a **finite-dimensional realization (FDR)**

- We say that the model possesses an FDR if there exists a mapping

$$G : R^d \to \mathcal{H}$$

$$\begin{aligned} dZ_t &= a(Z_t)dt + b(Z_t)dW_t, \quad Z_0 = z_0 \\ r_t(x) &= G(Z_t, x). \end{aligned}$$

Solutions...

- No non-trivial interest rate model consistent with Nelson-Siegel.
- There exists a model for Svensson family (quite limited one, since all the parameters (out of 6) but one have to be kept constant or deterministic).
- **However**, **good news** is that by constructing an FDR, we can find a minimal extension of the family which is consistent with a given interest rate model!
- Previous literature shows how this procedure has been done for a one-country model.

The following questions arise...

- To simplify estimation and pricing procedures it would be convenient to construct parameterized smooth families of forward rates for **two** country interest rate markets.

- How do we construct **consistent** families on international market?

- It is obvious that there is a connection between the two families.

- And related question - how and when is it possible to construct an FDR?

- We must take into account an exchange rate dynamics!!!

Existence of an FDR. Spot exchange rate

In the following cases we can derive **necessary and sufficient** conditions for an FDR to exist:

- Forward rate volatilities σ and $\tilde{\sigma}$ are deterministic

$$\sigma(\hat{r}, x) = \sigma(x), \quad \tilde{\sigma}(\hat{r}, x) = \tilde{\sigma}(x)$$

$$\delta(\hat{r}, x) = \delta$$

- Forward rate volatilities and the exchange rate volatility are of the form

$$\sigma(\tilde{r}, r, x) = \varphi(r, \tilde{r})\lambda(x) \quad \tilde{\sigma}(\tilde{r}, r, x) = \varphi(r, \tilde{r})\tilde{\lambda}(x)$$

$$\delta(\tilde{r}, r) = \delta\varphi(r, \tilde{r})$$

Quasi-exponential functions (QE)

A function is **quasi-exponential** if and only if it is a component of the solution of a vector valued linear Ordinary Differential Equation (ODE) with constant coefficients

$$f(x) = \sum_i e^{\lambda_i x} + \sum_j e^{\alpha_j x}[p_j(x)\cos(w_j x) + q_j(x)\sin(w_j x)]$$

whereas p_j and q_j are real polynomials.

Results. Spot exchange rate (cont.)

- The two-country model admits an FDR if and only if σ and $\tilde{\sigma}$ are quasi-exponential
- Existence of an FDR ⟹ Finite Dimension of the Lie algebra generated by $\hat{\mu}, \sigma_1, \ldots, \sigma_m$
- In deterministic volatility case the dimension of the relevant Lie algebra

$$dim\{\hat{\mu}, \hat{\sigma}_1, \hat{\sigma}_2, \ldots, \hat{\sigma}_m\}_{LA} \leq 1 + m + \sum_{i=1}^{m} n_i$$

where n_i is degree of polynomial M^i, such that

$$M^i(\mathbf{F})\sigma_{\mathbf{i}}(\mathbf{x}) = \mathbf{M^i}(\mathbf{F})\tilde{\sigma}_{\mathbf{i}}(\mathbf{x}) = \mathbf{0}$$

- **Volatility of the spot exchange rate does not play a role!!!**

Smaller dimension of the state space

- There exists an FDR with a **smaller** state space representation if and only if for each *i* the exchange rate volatility δ and the forward rate volatilities satisfy

$$\mathbf{B}M_1^i(\mathbf{F})\,(\sigma_i - \tilde{\sigma}_i) + M^i(0)\delta_i = 0, \qquad M^i(0) \neq 0,$$

where if M is defined as

$$M(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

$$M_1(x) = x^{n-1} + a_{n-1}x^{n-2} + \ldots + a_2 x + a_1$$

- **The exchange rate volatility plays a role !!!**

Results for the system with a forward exchange rate

Assume that the domestic and foreign forward rate volatilities, as well as a forward exchange rate volatility are deterministic, i.e. of the form

$$\sigma(\hat{r}, x) = \sigma(x), \quad \tilde{\sigma}(\hat{r}, x) = \tilde{\sigma}(x), \quad \delta(\hat{r}, x) = \delta(x)$$

Then there exists a finite dimensional realization if and only if both forward interest rate volatilities are **quasi-exponential**.

Forward exchange rate volatility can be expressed as

$$\delta(x) = \int_0^x \sigma(t, s) ds - \int_0^x \tilde{\sigma}(t, s) ds + \delta_S(t)$$

and thus also **quasi-exponential.**

Construction of an FDR

From the literature we know that if the Lie algebra $\{\hat{\mu}, \hat{\sigma}\}_{LA}$ is spanned by the smooth vector fields

$$f_1, \ldots, f_d$$

then for the initial point $\hat{r}^0$ all forward rate curves produced by the model will belong to the manifold, which can be parameterized as

$$\hat{G}(z_1, \ldots, z_d) = e^{f_d z_d} \ldots e^{f_1 z_1} \hat{r}^0$$

And where operator $e^{f_i z_i}$ is defined as a solution to

$$\begin{cases} \frac{\partial \hat{r}_t}{\partial t} & = & f(\hat{r}_t) \\ \hat{r}_0 & = & \hat{r}. \end{cases}$$

That is, integral curve for the vector field *f* passing through $\hat{r}^0$

Construction. Results

- In the simpler cases we can construct **minimal** realizations

- Since in the general constant direction volatility case we can study only enlarged Lie-algebras, we can obtain only **non-minimal** realizations (with excessive amount of parameters)

Examples. Choice of volatilities

- System with spot exchange rate

$$\sigma(x) = e^{-\alpha x}, \quad \tilde{\sigma}(x) = Ce^{-\alpha x}, \quad \delta(x) = \delta$$

$$\sigma(x) = e^{-\alpha x}, \quad \tilde{\sigma}(x) = e^{-2\alpha x}, \quad \delta(x) = \delta$$

- System with forward exchange rate

$$\sigma(x) = e^{-\alpha x}, \quad \tilde{\sigma}(x) = Ce^{-\alpha x}$$

$$\delta(x) = -\frac{1}{\alpha}(1 - C)e^{-\alpha x}$$

- Besides that we are free to choose the initial forward rate curves to be extended for both countries

Families of forward rates

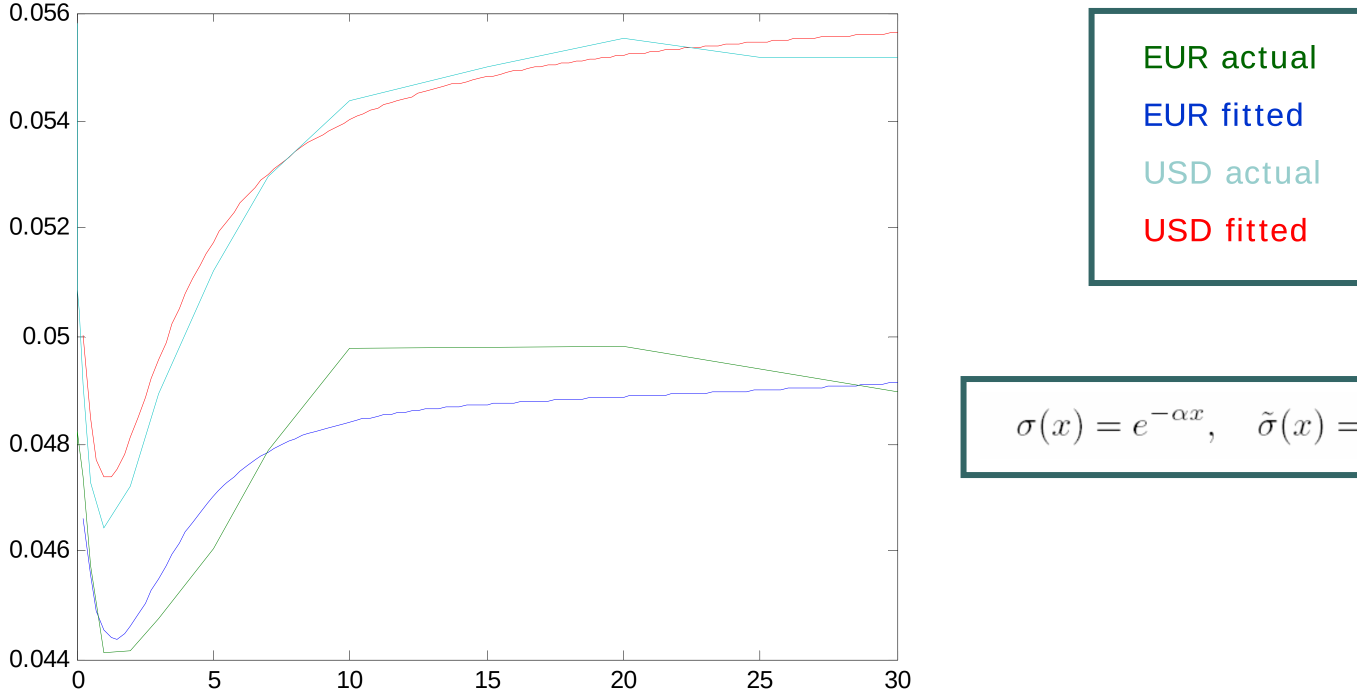
$$
\begin{aligned}
G_1(x) &= G_1^0(x) + z_1 e^{-\alpha x} + z_2 e^{-2\alpha x} - z_3 \alpha e^{-\alpha x} + z_4 e^{-\alpha x} \\
G_2(x) &= G_2^0(x) + z_1 C e^{-\alpha x} + z_2 C^2 e^{-2\alpha x} - C\alpha\delta z_2 e^{-\alpha x} - z_3 \alpha C e^{-\alpha x} + z_4 C^2 e^{-\alpha x} \\
G_3 &= G_3^0 + z^0 \left\{ \frac{1-C^2}{2\alpha^2} - \frac{\delta^2}{2} - \frac{C\delta}{\alpha} \right\} + \delta z_1 + \frac{1-C^2}{2\alpha} z_2 + (1-C) z_3 \\
&- \left\{ \frac{1-C^2}{\alpha} + \delta C \right\} z_4,
\end{aligned}
$$

$$\sigma(x) = e^{-\alpha x}, \quad \tilde{\sigma}(x) = C e^{-\alpha x}$$

$$
\begin{aligned}
G_1(x) &= G_1^0(x) + z_3 e^{-\alpha x} - z_4 e^{-2\alpha x} + z_0 e^{-\alpha x} - \alpha z_1 e^{-\alpha x} + \alpha^2 z_2 e^{-\alpha x} \\
G_2(x) &= G_2^0(x) + \frac{1}{2} z_4 e^{-2\alpha x} - \delta\alpha z_4 e^{-2\alpha x} - z_5 e^{-4\alpha x} + z_0 e^{-2\alpha x} - 2\alpha z_1 e^{-2\alpha x} \\
&+ 4\alpha^2 z_2 e^{-2\alpha x} \\
G_3 &= z^0 \left\{ \frac{3}{8\alpha^2} - \frac{\delta^2}{2} - \frac{\delta}{2\alpha} \right\} - \frac{z_3}{\alpha} + \left\{ \frac{3}{4\alpha} + \frac{\delta}{2} \right\} z_4 - \frac{1}{4\alpha} z_5 + \delta z_0 + \alpha z_2
\end{aligned}
$$

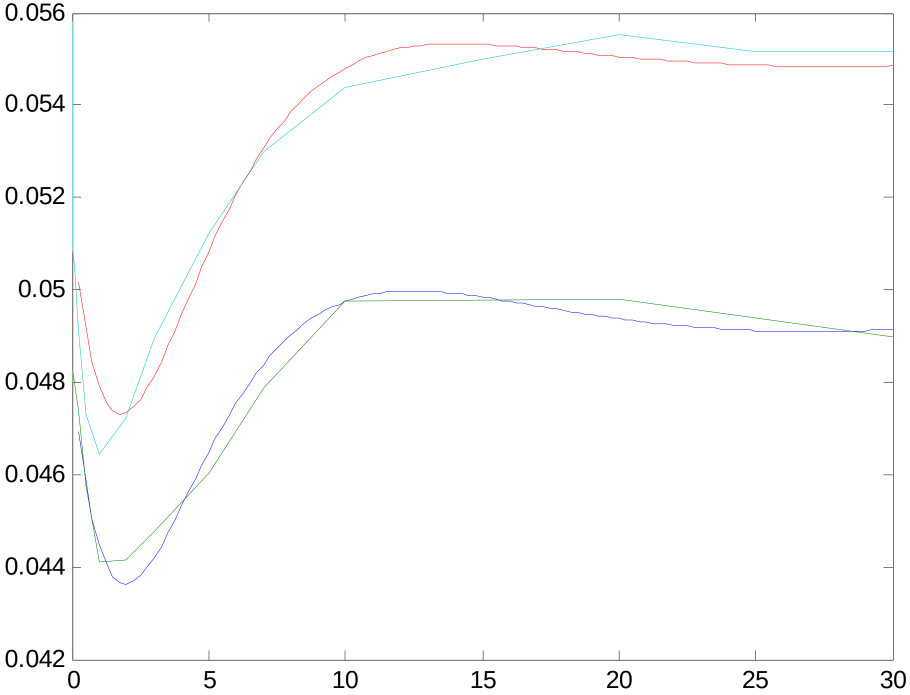
$$\sigma(x) = e^{-\alpha x}, \quad \tilde{\sigma}(x) = e^{-2\alpha x}$$

Estimation with the spot exchange rate



$$\sigma(x) = e^{-\alpha x}, \quad \tilde{\sigma}(x) = Ce^{-\alpha x}$$

Estimation with the spot exchange rate



EUR actual

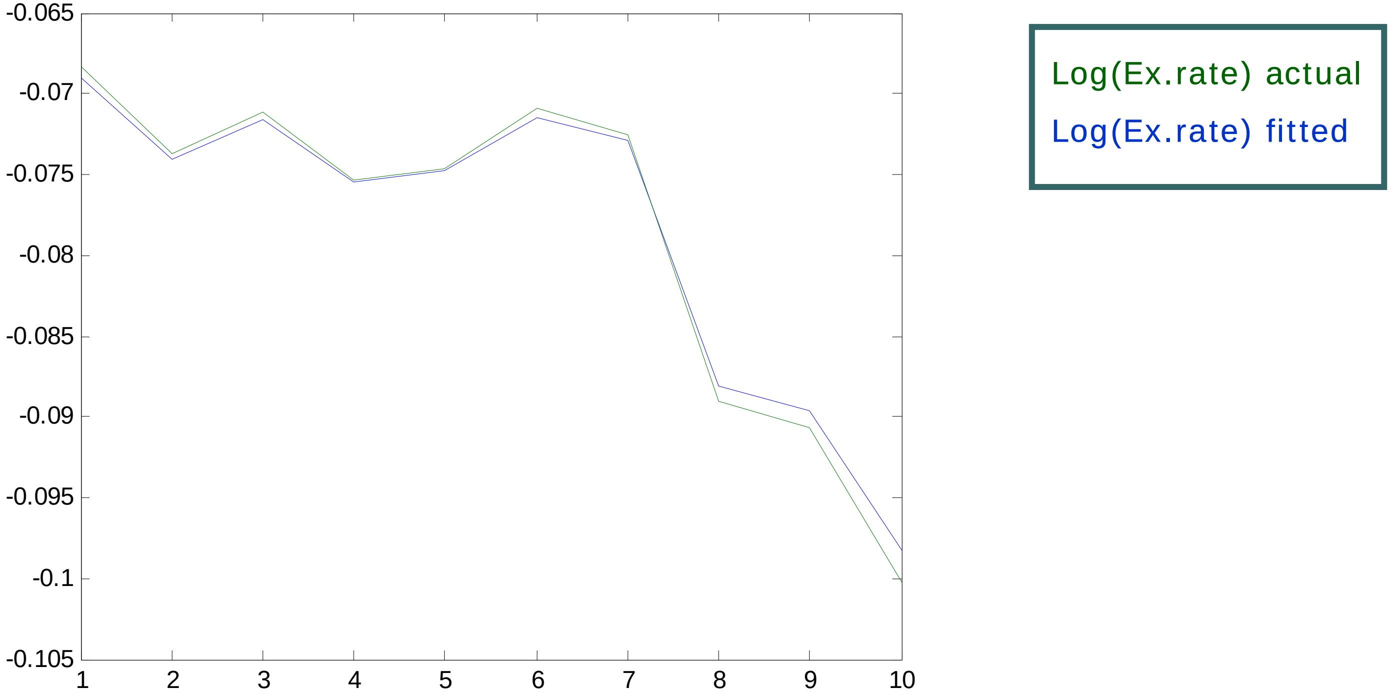
EUR fitted

USD actual

USD fitted

$$\sigma(x) = e^{-\alpha x}, \quad \tilde{\sigma}(x) = e^{-2\alpha x}$$

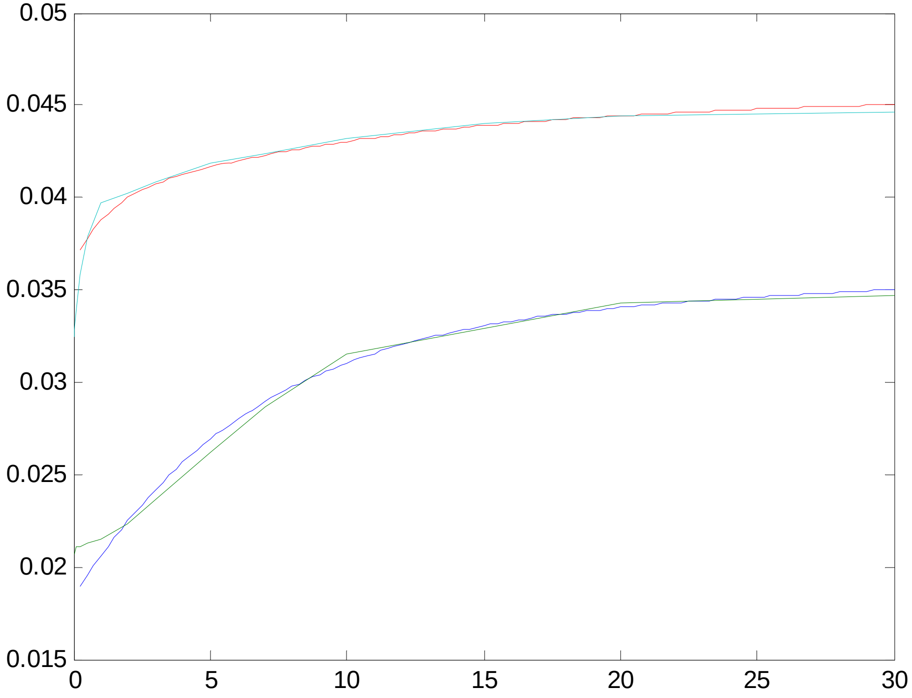
Case 2. Spot exchange rate



Families of forward interest rates and forward exchange rate

$$
\begin{aligned}
G_1(x) &= G_1^0(x) + k_1\alpha e^{-\alpha x} - k_2\alpha e^{-2\alpha x} + k_3\alpha e^{-\alpha x} \\
G_2(x) &= G_2^0(x) + k_1\alpha e^{-\alpha x}(C^2 - \delta C\alpha) - k_2\alpha e^{-2\alpha x} + k_3 C\alpha e^{-\alpha x} \\
G_3(x) &= G_3^0(x) - k_1\frac{C(1-C)}{\alpha}(1-e^{-\alpha x}) + k_2\frac{C(1-C)}{2\alpha}(1-e^{-2\alpha x}) \\
&- k_3(1-C)e^{-\alpha x},
\end{aligned}
$$

System with forward exchange rate 1



EUR actual

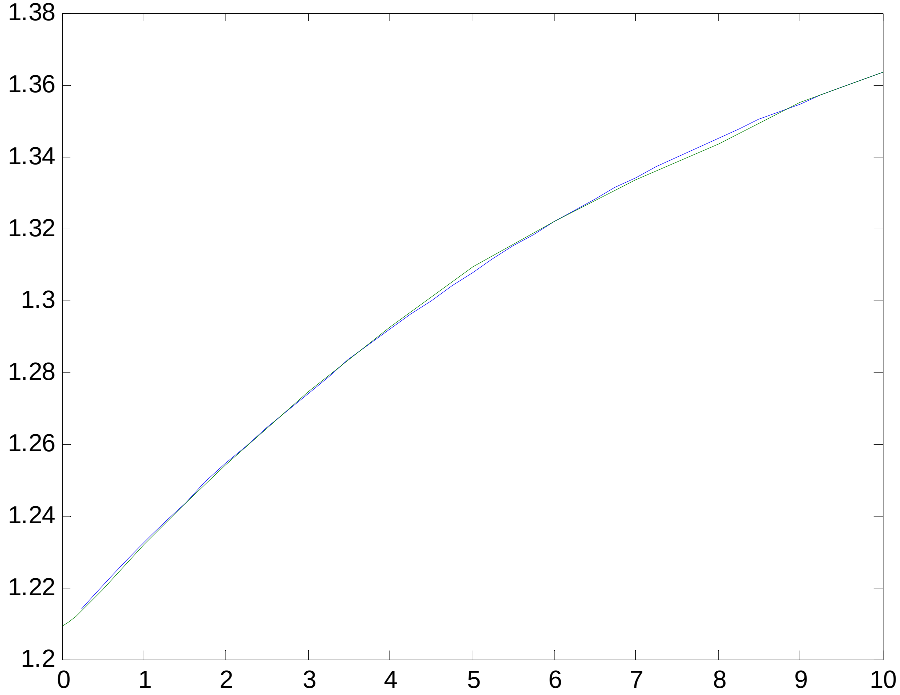
EUR fitted

USD actual

USD fitted

$$\sigma(x) = e^{-\alpha x}, \quad \tilde{\sigma}(x) = Ce^{-\alpha x}$$

System with forward exchange rate 1

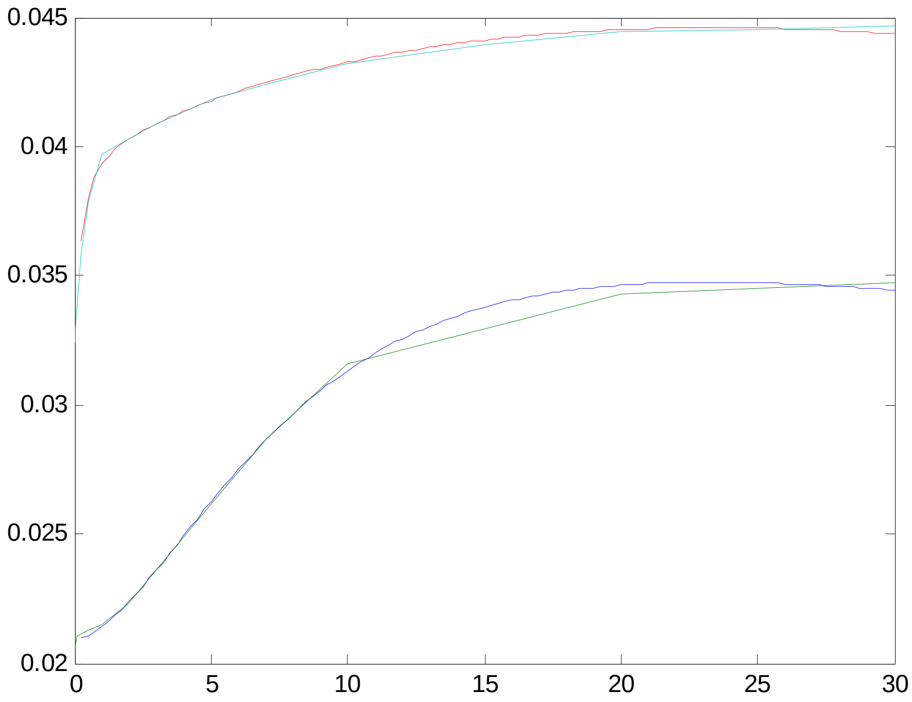


Forward ex.rate actual

Forward ex.rate fitted

$$\delta(x) = -\frac{1}{\alpha}(1 - C)e^{-\alpha x}$$

System with forward exchange rate 2



EUR actual

EUR fitted

USD actual

USD fitted

$$\sigma(x) = e^{-\alpha x}, \quad \tilde{\sigma}(x) = Ce^{-\alpha x}$$

Exchange rate is not involved in the estimation!!!

Future research

- We will allow all the objects to be driven by a multidimensional Wiener process as well as by a Levy process.

- We will consider other derivatives to see how the model performs.