Portfolio Optimisation with Economic Factors and Transaction Costs

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Outline

- The model
- Portfolio problems: discussion, examples
- Details: discounted reward (probem I)
- Details: average rate of return (problem II)

Financial market

asset prices $\in (0,\infty)^d$

$$\begin{pmatrix} S(t) \\ X(t) \end{pmatrix}_{t\in[0,\infty)}$$ – Markov (Feller) process

economic factors $\in \mathbb{R}^m$

$$S(t) = \big(S^1(t), S^2(t), \ldots, S^d(t)\big)$$

bank account $\equiv 1$

Transaction costs

$n_0 \in [0, \infty)^d$ – number of shares before transaction
$n_1 \in [0, \infty)^d$ – number of shares after transaction
$s_0 \in (0, \infty)^d$ – stock prices
$K \in (0, \infty)$, $k \in [0, 1)^d$ – transaction costs

$$c(n_0, n_1, s_0) = K + (n_1 - n_0)^T diag(k) s_0$$

$$c(n_0, n_1, s_0) = \max\left(K, (n_1 - n_0)^T diag(k) s_0\right)$$

$$\Downarrow$$

Impulsive portfolios

Selection of optimal portfolios (I)

$$\mathbb{E}\left\{ \int_0^\infty e^{-\alpha t} \; F\big(N(t), S(t), X(t)\big)\, dt + \sum_{k=1}^{\infty} e^{-\alpha \tau_k} G(C_k) \right\} \longrightarrow \max$$

share holding process (→ $N(t)$)

consumption (→ C_k)

$\Downarrow$

- banking and cash management
- tracking benchmark
- diversification
- risk management

Optimal portfolio selection: $F \equiv 0$

$$\mathbb{E}\left\{\sum_{k=1}^{\infty} e^{-\alpha\tau_k} G(C_k)\right\} \longrightarrow \max$$

- J.F. Eastham, K.J. Hastings, *Optimal impulse control of portfolios*, 1988, Math. Oper. Res. 13.4, pp. 588-605
- R. Korn *Portfolio optimisation with strictly positive transaction costs*, 1998, Finance and Stochastics, 2, pp. 85-114
- A.J. Morton, S.R. Pliska, *Optimal portfolio management with fixed transaction costs*, 1995, Mathematical Finance 5, pp. 337-356

Optimal portfolio selection: $G \equiv 0$

$$\mathbb{E}\left\{\int_0^\infty e^{-\alpha t} F\big(N(t), S(t), X(t)\big)\, dt\right\} \longrightarrow \max$$

- G.M. Constantinides, S.F. Richards *Existence of optimal simple policies for discounted-cost inventory and cash management in continuous time*, 1978, Oper. Res. 26.4, pp. 620-636
- A. Vollert *A Stochastic Control Framework for Real Options in Strategic Valuation*, 2003, Birkhäuser
- S.R. Pliska, K. Suzuki, *Optimal tracking for asset allocation with fixed and proportional transaction costs*, 2004, Quantitative Finance 4.2, pp. 233-243

Example A

$$\mathbb{E}\left\{ \int_0^\infty e^{-\alpha t} F\big(N(t), S(t), X(t)\big)\, dt + \sum_{k=1}^\infty e^{-\alpha \tau_k} G(C_k) \right\}$$

G - utility function

$F(\eta, s, x) = u(\eta \cdot s), \qquad u$ – utility function

Example B (proportion tracking)

$$\mathbb{E}\left\{ \int_0^\infty e^{-\alpha t} F\big(N(t), S(t), X(t)\big)\ dt + \sum_{k=1}^{\infty} e^{-\alpha \tau_k} G(C_k) \right\}$$

$G -$ utility function,

$$F(\eta, s, x) = (\eta \cdot s) \sum_{i=1}^{d} \alpha_i (1 - |w^i - w^{*i}|),$$

where

$$w^i = \frac{\eta^i s^i}{\eta \cdot s}, \qquad \alpha_i \geq 0, \qquad w^* \text{ - target proportion}$$

Selection of optimal portfolios (II)

portfolio wealth

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}\,\ln\big(W(T)\big) \longrightarrow \max$$

Well-established between economists:

$$W(t) \approx W(0)e^{\mu t} \quad \Longrightarrow \quad \frac{1}{T}\ln W(T) = \frac{1}{T}\Big(\ln W(0) + \mu T\Big) \longrightarrow \mu$$

rate of return

Expected rate of return

- M. Akian, A. Sulem, M. Taksar, *Dynamic optimisation of long term growth rate for a portfolio with transaction costs – the logarithmic utility case*, 2001, Mathematical Finance 11.2, pp. 153-188
- G. Iyengar, *Universal investemt in the markets with transaction costs*, 2005, Mathematical Finance 15.2, pp. 359-371
- Ł. Stettner, *Discrete time risk sensitive portfolio optimisation with consumption and proportional transaction costs*, 2005, Applicationes Mathematicae 32.4, pp. 395-404
- T. Tamura, *Maximizing the Growth Rate of a Portfolio with Fixed and Proportional Transaction Costs*, 2006, Applied Mathematics and Optimization 54, pp. 95-116

PROBLEM (I)

$$\mathbb{E}\left\{ \int_0^\infty \quad e^{-\alpha t} \quad F\Big(N(t), S(t), X(t)\Big)\ dt + \sum_{k=1}^\infty \quad e^{-\alpha \tau_k} \quad G(C_k) \right\}$$

PROBLEM (I)

$$\mathbb{E}\left\{ \int_0^\infty e^{-\alpha t} F\Big(N(t), S(t), X(t)\Big)\, dt + \sum_{k=1}^{\infty} e^{-\alpha \tau_k}\, G(C_k) \right\}$$

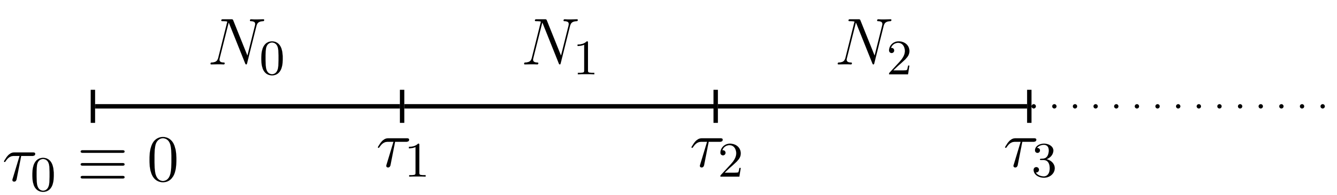
discount factor

Admissible strategies $\mathcal{A}$

$$\Pi = \big((N_0, 0), (N_1, C_1, \tau_1), (N_2, C_2, \tau_2), \dots\big)$$

number of shares after transaction (no short-sales) — N_1

transaction time — τ_1



τ_i – stopping time,

$$N_i \in L^0(\Omega, \mathcal{F}_{\tau_i}, \mathbb{P}; [0, \infty)^d)$$

$$C_i \in L^0(\Omega, \mathcal{F}_{\tau_i}, \mathbb{P}; [0, \infty))$$

Admissible strategies $\mathcal{A}$, contd.

- Self-financing condition

$$N_i \cdot S(\tau_i) + C_i + c\big(N_{i-1}, N_i, S(\tau_i)\big) = N_{i-1} \cdot S(\tau_i)$$

$$\sum_{k=1}^{d} N_i^k S^k(\tau_i)$$ consumption transaction costs

- Separation of transactions

$$\mathbb{P}(\lim_{i \to \infty} \tau_i = \infty) = 1$$

- $\mathcal{A} = \mathcal{A}(s_0, x_0)$, (s_0, x_0) - starting point of the price/factors process

Value function

$$v(n_0, s_0, x_0) = \sup_{\substack{\Pi \in \mathcal{A}(s_0, x_0), \\ N^{\Pi}(0) = n_0}} J(\Pi)$$

reward functional

$$J(\Pi) := \mathbb{E}^{\,s_0, x_0} \left\{ \int_0^\infty e^{-\alpha t} F\big(N^{\Pi}(t), S(t), X(t)\big)\, dt + \sum_{k=1}^{\infty} e^{-\alpha \tau_k^{\Pi}} G(C_k^{\Pi}) \right\}$$

Assumptions

CONT: F and G are continuous and non-negative

$$E = [0,\infty)^d \times (0,\infty)^d \times \mathbb{R}^m$$

NFL1: for any $y = (n,s,x) \in E$ there exists a ball $B(y,\epsilon) \subset E$, $\kappa > 1$, $\eta < \alpha$

$$\sup_{(n_0,s_0,x_0)\in B(y,\epsilon)} \; \sup_{\substack{\Pi \in \mathcal{A}(s_0,x_0),\\ N^{\Pi}(0)=n_0}} \mathbb{E}^{\,s_0,x_0}\left(N^{\Pi}(t)\cdot S(t)\right)^{\kappa} \approx O(e^{\eta t})$$

Assumptions contd.

NFL2: for any $y = (n, s, x) \in E$ there exists a ball $B(y, \epsilon) \subset E$ and $\kappa > 1$

$$\sup_{(n_0,s_0,x_0)\in B(y,\epsilon)} \sup_{\substack{\Pi \in \mathcal{A}(s_0, x_0), \\ N^{\Pi}(0) = n_0}}$$

$$\mathbb{E}^{s_0,x_0}\left\{ \int_0^\infty e^{-\alpha t}\Big(F\big(N^{\Pi}(t), S(t), X(t)\big)\Big)^{\kappa} dt + \sum_{k=1}^{\infty} e^{-\alpha \tau_k^{\Pi}} \big(G(C_k^{\Pi})\big)^{\kappa} \right\}$$

$$< \infty$$

Optimal strategy

Theorem. The value function v is continuous and satisfies an appropriate Bellman equation. Moreover, there exists an optimal portfolio:

$$\tau_i = \inf\{t > \tau_{i-1} : \ \big(N_{i-1}, S(t), X(t)\big) \in I\}$$

$$N_i = D_1\big(N_{i-1}, S(\tau_i), X(\tau_i)\big)$$

$$C_i = D_2\big(N_{i-1}, S(\tau_i), X(\tau_i)\big)$$

a measurable functions

impulse region $I \subset E$

Details

Multiplicative price process

$$\begin{pmatrix} Z(t) \\ X(t) \end{pmatrix} \in \mathbb{R}^d \times \mathbb{R}^m$$ – Feller process

$$S^i(t) = e^{Z^i(t)}, \quad i = 1, \dots d.$$

MLT: "conditionally exponential growth"

$$\mathbb{E}^{(z_0,x_0)}\Big(\big(S^i(t)\big)^2\big|\mathcal{F}_s\Big) \le \big(S^i(s)\big)^2 e^{2\beta(t-s)}, \quad 0 \le s \le t$$

Multiplicative price process, contd.

AF: $\exists A, B \geq 0 \quad 0 \leq F(n_0, s_0, x_0) \leq A + B(n_0 \cdot s_0)$

MLT:

$$\mathbb{E}^{(z_0,x_0)}\Big(\big(S^i(t)\big)^2\big|\mathcal{F}^s\Big) \leq \big(S^i(s)\big)^2 e^{2\beta(t-s)}, \quad 0 \leq s \leq t$$

Theorem. For $\alpha > \beta$:

- Bounded F, G + MLT $\Rightarrow$ optimal strategy
- Unbounded F + bounded G + AF + MLT $\Rightarrow$ optimal strategy

Diffusion with bounded coefficients

$$S^1(t) = 1$$

$$\frac{dS^k(t)}{S^k(t)} = \mu^k\big(X(t)\big)dt + \sigma^k\big(X(t)\big)\cdot dW(t), \qquad k = 2, \ldots, d,$$

$\mu^k : \mathbb{R}^m \to \mathbb{R}$ $\qquad$ $\sigma^k : \mathbb{R}^m \to \mathbb{R}^p$

Brownian motion in $\mathbb{R}^p$

$X(t) \in \mathbb{R}^m$ – Feller process

Diffusion, contd.

Notice that MLT is satisfied with

$$\beta = \sup_{k=1,\dots,d} \Big(\frac{1}{2} \sup_{x \in \mathbb{R}^m} \|\sigma^k(x)\|^2 + \sup_{x \in \mathbb{R}^m} |\mu^k(x)|\Big),$$

but we can obtain more.

Diffusion, contd.

AF: $\exists A, B \geq 0 \quad 0 \leq F(n_0, s_0, x_0) \leq A + B(n_0 \cdot s_0)$

AG: $\exists A, B \geq 0 \quad 0 \leq G(c) \leq A + Bc$

Theorem. Unbounded F, G + AF + AG $\Rightarrow$ optimal strategy for $\alpha > \tilde{\beta}$,

$$\tilde{\beta} = \frac{1}{2} \sup_{k=1,\ldots,d} \Big(\sup_{x \in \mathbb{R}^m} \|\sigma^k(x)\|^2 + \sup_{x \in \mathbb{R}^m} |\mu^k(x)| \Big)$$

PROBLEM (II)

$$\liminf_{T \to \infty} \frac{1}{T} \mathbb{E} \ln \big(W(T) \big) \longrightarrow \max$$

Restrictions

- no consumption
- discrete time
- transaction costs:

small enough

$$c(n_0, n_1, s_0) = K + (n_1 - n_0)^T diag(k) s_0$$

- price process:

$$\frac{S^i(t+1)}{S^i(t)} = \zeta^i\big(X(t+1), \xi(t+1)\big), \quad i = 1, \ldots, d$$

i.i.d. random variables independent of $\big(X(t)\big)_{t \geq 0}$

Assumptions

A1: $\big(S(t), X(t)\big)$ is a Feller process

A2: The function

$\sum \pi^i = 1, \; \pi^i \geq 0$

$$(\pi, x) \mapsto \mathbb{E}^{\,x}\Big\{ \ln\Big(\pi \cdot \zeta\big(X(1), \xi(1)\big)\Big)\Big\}$$

is bounded and continuous

A3: For some $n \geq 1$

$$\sup_{x,x' \in \mathbb{R}^m} \; \sup_{B \in \mathcal{B}(\mathbb{R}^m)} \Big(P^n(x, B) - P^n(x', B) \Big) = \kappa < 1$$

transition operator of the process $X(t)$

Assumption A1

$X(t)$ – Feller process
$x \mapsto \zeta^i(x, \xi)$ – continuous function $\Longrightarrow$ **A1**

Diffusion with economic factors

Wiener process in $\mathbb{R}^p$ (→ W)

$$\frac{S^i(t+1)}{S^i(t)} = \exp\Big(\sigma^i\big(X(t+1)\big)\cdot\big(W(t+1)-W(t)\big)+\mu^i\big(X(t+1)\big)\Big),$$

$$i = 1, \ldots, d$$

σ^i, μ^i – continuous with values in $\mathbb{R}^p, \mathbb{R}$, resp.

Assumption A2

A2: $(\pi, x) \mapsto \mathbb{E}^x\{\ln \pi \cdot \zeta\big(X(1), \xi(1)\big)\}$ is bounded and continuous

- $\zeta^i(x, \xi)$ is bounded, separated from 0, continuous in x
- Diffusion with economic factors:

$$\frac{S^i(t+1)}{S^i(t)} = \exp\Big(\sigma^i\big(X(t+1)\big) \cdot \big(W(t+1) - W(t)\big) + \mu^i\big(X(t+1)\big)\Big)$$

μ^i, σ^i continuous and bounded.

Assumption A3

A3: For some $n \geq 1$

$$\sup_{x,x' \in \mathbb{R}^m} \sup_{B \in \mathcal{B}(\mathbb{R}^m)} \Big(P^n(x,B) - P^n(x',B) \Big) = \kappa < 1$$

- widely used with $n = 1$ and called Uniform Ergodicity,
- often satisfied by Markov chains with finite state space,
- implies existence of a unique invariant probability measure

Optimal portfolio

Theorem. Under (A1)-(A3) there exists a constant λ such that for any $(n_0, s_0, x_0) \in E$

$$\sup_{\substack{\Pi \in \mathcal{A}(s_0, x_0), \\ N^\Pi(0) = n_0}} \liminf_{T \to \infty} \frac{1}{T} \mathbb{E}^{\,s_0, x_0} \ln \big(N^\Pi(T) \cdot S(T) \big) = \lambda.$$

This supremum is attained by the portfolio Π^* given by

$$\tau_i = \inf\{t > \tau_{i-1} : \ \big(N_{i-1}, S(t), X(t)\big) \in I\}$$

$$N_i = D\big(N_{i-1}, S(\tau_i), X(\tau_i)\big)$$

a measurable function

impulse region $I \subset E$

Comments

- the portfolio is also optimal for the problem with only proportional transaction costs (the optimal value is still equal to λ)

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  - the constant term of the transaction costs spoils continuity
  - Vanishing Discount Approach leading to Bellman inequality
  - Tauberian theorem
  - Large Deviations for estimates

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  - the constant term of the transaction costs spoils continuity
  - Vanishing Discount Approach leading to Bellman inequality
  - Tauberian theorem
  - Large Deviations for estimates
- Bellman equality requires further assumptions
- incomplete observation of economic factors

Details

Bellman equation

- switching functional: selects the best possible transaction

$$M\psi(n_0, s_0, x_0) = \sup\Big\{\psi(n_1, s_0, x_0) + G(c_1) : \\ n_1 \cdot s_0 + c_1 + c(n_0, n_1, s_0) = n_0 \cdot s_0\Big\} \\ \vee\, \psi(0, s_0, x_0)$$

- Bellman equation

$$v(n_0, s_0, x_0) = \\ \sup_\tau \mathbb{E}^{(s_0,x_0)}\bigg(\int_0^\tau e^{-\alpha t} F\big(N^\Pi(t), S(t), X(t)\big)\, dt \\ + e^{-\alpha\tau} Mv\big(n_0, S(\tau), X(\tau)\big)\bigg)$$

Optimal strategy

Optimal strategy:

- impulse region

$$I = \{(n,s,x) : v(n,s,x) = Mv(n,s,x)\}$$

- strategy

$$\tau_i = \inf\{t > \tau_{i-1} : \ \big(N_{i-1}, S(t), X(t)\big) \in I\},$$
$$N_i = D_1\big(N_{i-1}, S(\tau_i), X(\tau_i)\big),$$
$$C_i = D_2\big(N_{i-1}, S(\tau_i), X(\tau_i)\big).$$

($D = (D_1, D_2)$ – a measurable selector for Mv

Proof

v_0 – a value function if no transactions are allowed
v_i – a value function if at most i transactions are allowed
v – a value function for the main problem

Lemma. Functions v_i are continuous and $v_i \uparrow v$.

Goal: prove that this convergence is uniform on compacts

Main observation: for $\Pi \in \mathcal{A}(s_0, x_0)$ there exists a doubled strategy $\tilde{\Pi} \in \mathcal{A}(s_0, x_0)$ such that

- $2N_0^{\Pi} = N_0^{\tilde{\Pi}}$ and $2N_i^{\Pi} \leq N_i^{\tilde{\Pi}}$, $i \geq 1$,
- $\tau_i^{\Pi} = \tau_i^{\tilde{\Pi}}$, $i \geq 1$,
- savings on transaction costs are invested in one of the assets