Bounds for Functions of Multivariate Risks

Bounding the Value-at-Risk for an aggregate risk

Paul Embrechts[1] Giovanni Puccetti[2]

[1]Department of Mathematics ETH Zurich, CH-8092 Zurich, Switzerland

[2]Department of Mathematics for Decisions, University of Firenze, 50134 Firenze, Italy

Journal of Multivariate Analysys, in press

The problem at hand

We consider an insurance company holding a portfolio

$$X := (X_1, \ldots, X_n)$$

of n one-period risks on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

Typically, the statistics gathered by the insurer give information about the **marginal** distribution functions (dfs) of the risks,

$$F_1, \ldots, F_n,$$

but not about their **joint df**, i.e. the way the risks are **interrelated**.

Given a measurable **non-decreasing** function $\psi : \mathbb{R}^n \to \mathbb{R}$, the aggregate loss which the insurer will bear is

$$\psi(X) = \psi(X_1, \ldots, X_n).$$

The problem at hand

We consider an insurance company holding a portfolio

$$X := (X_1, \ldots, X_n)$$

of n one-period risks on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

Typically, the statistics gathered by the insurer give information about the **marginal** distribution functions (dfs) of the risks,

$$F_1, \ldots, F_n,$$

but not about their **joint df**, i.e. the way the risks are **interrelated**.

Given a measurable **non-decreasing** function $\psi : \mathbb{R}^n \to \mathbb{R}$, the aggregate loss which the insurer will bear is

$$\psi(X) = \psi(X_1, \ldots, X_n).$$

The problem at hand

We consider an insurance company holding a portfolio

$$X := (X_1, \dots, X_n)$$

of n one-period risks on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

Typically, the statistics gathered by the insurer give information about the **marginal** distribution functions (dfs) of the risks,

$$F_1, \dots, F_n,$$

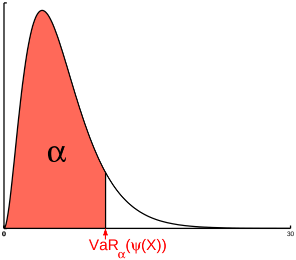
but not about their **joint df**, i.e. the way the risks are **interrelated**.

Given a measurable **non-decreasing** function $\psi : \mathbb{R}^n \to \mathbb{R}$, the aggregate loss which the insurer will bear is

$$\psi(X) = \psi(X_1, \dots, X_n).$$

Value-at-Risk for the aggregate loss

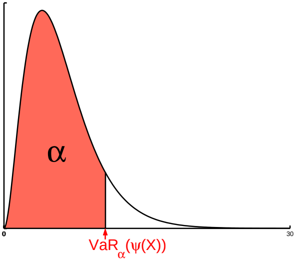
The Value-at-Risk at probability level α for $\psi(X)$ is the maximum aggregate loss which can occur with probability $\alpha, \alpha \in [0, 1]$.



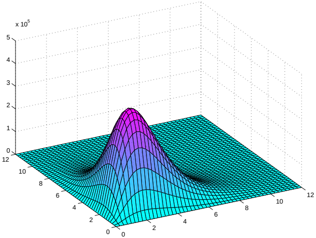
Calculating all the VaRs (quantiles) for the aggregate loss $\psi(X)$ is equivalent to inverting its distribution function $F(x) := \mathbb{P}[\psi(X) < x], x \in \mathbb{R}$.

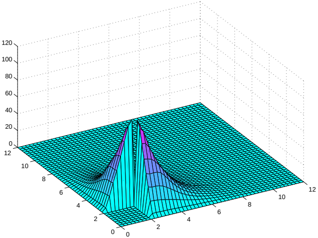
Value-at-Risk for the aggregate loss

The Value-at-Risk at probability level α for $\psi(X)$ is the maximum aggregate loss which can occur with probability α, $\alpha \in [0,1]$.



Calculating all the VaRs (quantiles) for the aggregate loss $\psi(X)$ is equivalent to inverting its distribution function
$F(x) := \mathbb{P}[\psi(X) < x], x \in \mathbb{R}$.

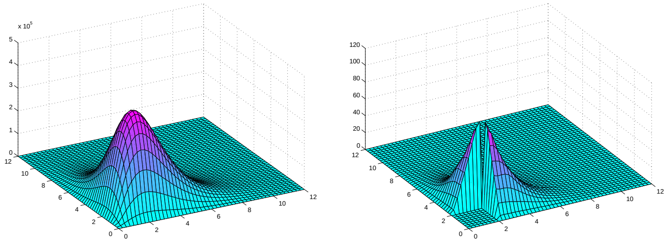




The distribution function F for the aggregate loss $\psi(X)$ cannot be determined without further information.

Moreover, note that there exists a df for $\psi(X) = \sum_{i=1}^n X_i$ having the given marginals $F_1, \ldots, F_n$ such that

$$\sum_{i=1}^{n} \mathrm{VaR}_\alpha (X_i) < \mathrm{VaR}_\alpha \left(\sum_{i=1}^{n} X_i \right)$$



The distribution function F for the aggregate loss $\psi(X)$ cannot be determined without further information.
Moreover, note that there exists a df for $\psi(X) = \sum_{i=1}^{n} X_i$ having the given marginals $F_1, \ldots, F_n$ such that

$$\sum_{i=1}^{n} \mathrm{VaR}_\alpha(X_i) < \mathrm{VaR}_\alpha\left(\sum_{i=1}^{n} X_i\right)$$

Mathematical problems with univariate marginals

Therefore, we have to search for the worst-possible $\mathrm{VaR}_\alpha(\psi(X))$ on

$$\mathfrak{F}(F_1, \ldots, F_n),$$

the set of dfs having $F_1, \ldots, F_n$ as fixed marginals.

Since bounding the VaR for the aggregate loss means bounding its distribution (tail) function from below (above), the problem at hand becomes determining

$$m_\psi(s) := \inf\{\mathbb{P}[\psi(X_1, \ldots, X_n) < s] : X_i \backsim F_i, 1 \leq i \leq n\}, s \in \mathbb{R},$$
$$M_\psi(s) := \sup\{\mathbb{P}[\psi(X_1, \ldots, X_n) \geq s] : X_i \backsim F_i, 1 \leq i \leq n\}, s \in \mathbb{R},$$

for a function $\psi : \mathbb{R}^n \to \mathbb{R}$.

Mathematical problems with multivariate marginals

Therefore, we have to search for the worst-possible $\mathrm{VaR}_\alpha(\psi(X))$ on

$$\mathfrak{F}(F_1, \ldots, F_n),$$

the set of dfs having $F_1, \ldots, F_n$ as fixed marginals.

Since bounding the VaR for the aggregate loss means bounding its distribution (tail) function from below (above), the problem at hand becomes determining

$$m_\psi(\vec{s}) := \inf\{\mathbb{P}[\psi(\vec{X}_1, \ldots, \vec{X}_n) < \vec{s}] : \vec{X}_i \backsim F_i, 1 \leq i \leq n\}, \vec{s} \in \mathbb{R}^k,$$
$$M_\psi(\vec{s}) := \sup\{\mathbb{P}[\psi(\vec{X}_1, \ldots, \vec{X}_n) \geq \vec{s}] : \vec{X}_i \backsim F_i, 1 \leq i \leq n\}, \vec{s} \in \mathbb{R}^k,$$

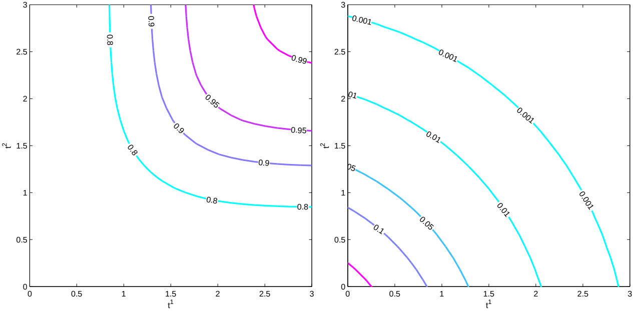
for a function $\psi : (\mathbb{R}^k)^n \to \mathbb{R}^k$.

Why working with multivariate marginals

Assuming multivariate marginals allows not only to fix the univariate df of every component of the single multivariate policies, but also the dependence **within** the single risks.

$$\begin{array}{r} \text{insurance line } 1 \rightarrow \\ \vdots \\ \text{insurance line } k \rightarrow \end{array} \quad \psi\left(\underbrace{\begin{pmatrix} X_1^1 \\ \vdots \\ X_1^k \end{pmatrix}}_{\text{policy } 1}, \dots, \underbrace{\begin{pmatrix} X_n^1 \\ \vdots \\ X_n^k \end{pmatrix}}_{\text{policy } n} \right) = \begin{pmatrix} X_1^1 + \cdots + X_n^1 \\ \vdots \\ X_1^k + \cdots + X_n^k \end{pmatrix}$$

With a multivariate aggregate loss, the definition of VaR does not make sense, since one should invert a distribution function $F : \mathbb{R}^n \to [0,1]$.



An intuitive and immediate measure of the risk involved in a multivariate loss df F is represented by the α-level sets of its df and of its tail $\overline{F}$.

Duality

$m_\psi(s)$ and $M_\psi(s)$ are two **linear problems** over a convex feasible space of measures. Therefore, they admit a **dual representation**.

Main Duality Theorem (Ramachandran and Rüschendorf (1995))

$$m_\psi(\vec{s}) = \sup \left\{ \sum_{i=1}^{n} \int_{\mathbb{R}^k} f_i dF_i : f_i \in L^1(F_i), i \in N \text{ with } \sum_{i=1}^{n} f_i(\vec{x}_i) \leq 1_{(-\infty,\vec{s})}(\psi(\vec{x})) \text{ for all } \vec{x} \in (\mathbb{R}^k)^n \right\},$$

$M_\psi(\vec{s})$ admits an analogous representation.

Duality

$m_\psi(s)$ and $M_\psi(s)$ are two **linear problems** over a convex feasible space of measures. Therefore, they admit a **dual representation**.

Main Duality Theorem (Ramachandran and Rüschendorf (1995))

$$m_\psi(\vec{s}) = \sup\left\{ \sum_{i=1}^n \int_{\mathbb{R}^k} f_i dF_i : f_i \in L^1(F_i), i \in N \textit{ with} \right.$$
$$\left. \sum_{i=1}^n f_i(\vec{x}_i) \le 1_{(-\infty,\vec{s})}(\psi(\vec{x})) \textit{ for all } \vec{x} \in (\mathbb{R}^k)^n \right\},$$

$M_\psi(\vec{s})$ *admits an analogous representation.*

Known solutions

$m_\psi(\vec{s})$ and $M_\psi(\vec{s})$, as well as their dual counterparts, are very difficult to solve. Solutions under general marginal dfs are known only in few cases.

- When $k = 1$ and $n = 2$; see Embrechts and Puccetti (2005).
- For $\psi = +$, Li, Scarsini, and Shaked (1996) give $m_\psi(\vec{s})$ for $n = 2$ and arbitrary k.
- When $n > 2$, the only explicit solution known is given in Rüschendorf (1982) for the sum of risks uniformly distributed on the unit interval and in our paper for the sum of risks uniformly distributed on the unit hypercube.

Known solutions

$m_\psi(\vec{s})$ and $M_\psi(\vec{s})$, as well as their dual counterparts, are very difficult to solve. Solutions under general marginal dfs are known only in few cases.

- When $k = 1$ and $n = 2$; see Embrechts and Puccetti (2005).
- For $\psi = +$, Li, Scarsini, and Shaked (1996) give $m_\psi(\vec{s})$ for $n = 2$ and arbitrary k.
- When $n > 2$, the only explicit solution known is given in Rüschendorf (1982) for the sum of risks uniformly distributed on the unit interval and in our paper for the sum of risks uniformly distributed on the unit hypercube.

Known solutions

$m_\psi(\vec{s})$ and $M_\psi(\vec{s})$, as well as their dual counterparts, are very difficult to solve. Solutions under general marginal dfs are known only in few cases.

- When $k = 1$ and $n = 2$; see Embrechts and Puccetti (2005).
- For $\psi = +$, Li, Scarsini, and Shaked (1996) give $m_\psi(\vec{s})$ for $n = 2$ and arbitrary k.
- When $n > 2$, the only explicit solution known is given in Rüschendorf (1982) for the sum of risks uniformly distributed on the unit interval and in our paper for the sum of risks uniformly distributed on the unit hypercube.

Known solutions

$m_\psi(\vec{s})$ and $M_\psi(\vec{s})$, as well as their dual counterparts, are very difficult to solve. Solutions under general marginal dfs are known only in few cases.

- When $k = 1$ and $n = 2$; see Embrechts and Puccetti (2005).
- For $\psi = +$, Li, Scarsini, and Shaked (1996) give $m_\psi(\vec{s})$ for $n = 2$ and arbitrary k.
- When $n > 2$, the only explicit solution known is given in Rüschendorf (1982) for the sum of risks uniformly distributed on the unit interval and in our paper for the sum of risks uniformly distributed on the unit hypercube.

The basic idea in the dual approach

If $\hat{\mathbf{f}} = (\hat{f}_1, \ldots, \hat{f}_n)$ and $\hat{\mathbf{g}} = (\hat{g}_1, \ldots, \hat{g}_n)$ are two set of functions which are admissible for the corresponding dual problems, we have

$$\mathbb{P}[\psi(\vec{X}) < \vec{s}] \geq m_\psi(\vec{s}) \geq \sum_{i=1}^{n} \int_{\mathbb{R}^k} \hat{f}_i dF_i,$$

$$\mathbb{P}[\psi(\vec{X}) \geq \vec{s}] \leq M_\psi(\vec{s}) \leq \sum_{i=1}^{n} \int_{\mathbb{R}^k} \hat{g}_i dF_i.$$

Therefore, even if we do not solve the dual problems, **dual admissible functions provide bounds on the solutions which are conservative from a risk management viewpoint**.

Dual bounds

We call *dual bounds* those bounds obtained by choosing *piecewise-linear* dual choices.

The dual bounds:

- **are better** than the bounds generally used in the literature; see Denuit, Genest, and Marceau (1999) and Embrechts, Höing, and Juri (2003).
- can be given also for non-homogemeous portfolios of risk; see Embrechts and Puccetti (2005)

Dual bounds

We call *dual bounds* those bounds obtained by choosing *piecewise-linear* dual choices.

The dual bounds:

- **are better** than the bounds generally used in the literature; see Denuit, Genest, and Marceau (1999) and Embrechts, Höing, and Juri (2003).
- can be given also for non-homogemeous portfolios of risk; see Embrechts and Puccetti (2005)

Dual bounds

We call *dual bounds* those bounds obtained by choosing *piecewise-linear* dual choices.

The dual bounds:

- **are better** than the bounds generally used in the literature; see Denuit, Genest, and Marceau (1999) and Embrechts, Höing, and Juri (2003).
- can be given also for non-homogemeous portfolios of risk; see Embrechts and Puccetti (2005)

Bounds on Value-at-Risk

	$\mathrm{VaR}_\alpha(\sum_{i=1}^{10} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{100} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{1000} X_i)$	
α	dual	standard	dual	standard	dual	standard
0.90	0.669	1.485	11.039	149.850	150.162	14998.500
0.95	1.353	2.985	22.227	229.850	301.823	29998.500
0.99	2.985	14.985	111.731	1499.850	1515.111	149998.500
0.999	68.382	149.985	1118.652	14999.850	15164.604	1499998.500

Table: Upper bounds for $\mathrm{VaR}_\alpha(\sum_{i=1}^{n} X_i)$ of three Pareto portfolios of different dimensions. Data in thousands.

Bounds on Value-at-Risk

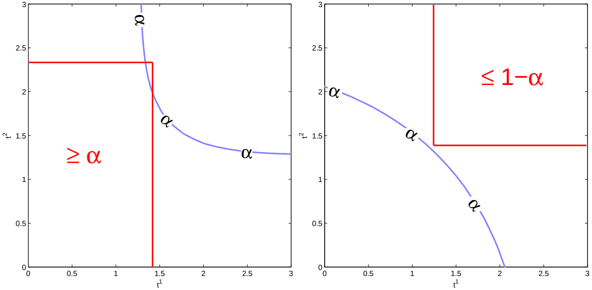
We can obtain the above table also for Moscadelli (2004)'s OR-portfolio.

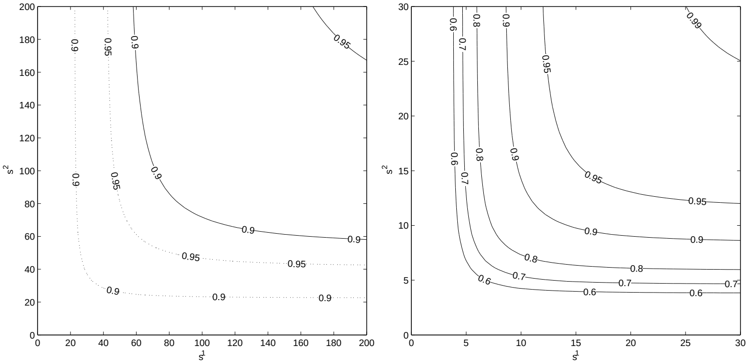
α	comonotonic value	dual bound	standard bound
0.99	2.8924×10^4	1.4778×10^5	2.6950×10^5
0.995	6.7034×10^4	3.3922×10^5	6.1114×10^5
0.999	4.8347×10^5	2.3807×10^6	4.1685×10^6
0.9999	8.7476×10^6	4.0740×10^7	6.7936×10^7

Table: Range for $\mathrm{VaR}_\alpha\left(\sum_{i=1}^{8} X_i\right)$ for the data underlying Moscadelli (2004).

Multivariate Value-at-Risk

The LO-VaR$_\alpha$ for m_ψ (left) and the UO-VaR$_\alpha$ for M_ψ (right) provide conservative estimates of the α-VaRs for the aggregate loss $\psi(\vec{X})$ over $\mathfrak{F}(F_1, \ldots, F_n)$.





Worst-possible LO-VaRs for the sum of two bivariate Pareto ($\theta = 1.2$ for the dotted line) (left) and Log-Normal (right) distributed risks.

Summary

Bounding the df for a non-decreasing function of dependent random vectors having fixed marginals

↓

general optimal solution is difficult to find when $n > 2$

↓

using the dual formulation **we can improve the standard bounds obtained from elementary probability**

↓

dual bounds can be computed for large heterogenous portfolios (operational risk)

Summary

Bounding the df for a non-decreasing function of dependent random vectors having fixed marginals

↓

general optimal solution is difficult to find when $n > 2$

↓

using the dual formulation **we can improve the standard bounds obtained from elementary probability**

↓

dual bounds can be computed for large heterogenous portfolios (operational risk)

Summary

Bounding the df for a non-decreasing function of dependent random vectors having fixed marginals

$\downarrow$

general optimal solution is difficult to find when $n > 2$

$\downarrow$

using the dual formulation **we can improve the standard bounds obtained from elementary probability**

$\downarrow$

dual bounds can be computed for large heterogenous portfolios (operational risk)

Summary

Bounding the df for a non-decreasing function of dependent random vectors having fixed marginals

$\downarrow$

general optimal solution is difficult to find when $n > 2$

$\downarrow$

using the dual formulation **we can improve the standard bounds obtained from elementary probability**

$\downarrow$

dual bounds can be computed for large heterogenous portfolios (operational risk)

Acknowledgements

The second author would like to thank

Marco Scarsini

for helpful discussions,

and

RiskLab, ETH Zurich

for financial support.

Acknowledgements

The second author would like to thank
Marco Scarsini
for helpful discussions,
and
RiskLab, ETH Zurich
for financial support.

For Further Reading I

Denuit, M., C. Genest, and É. Marceau (1999). Stochastic bounds on sums of dependent risks. *Insurance Math. Econom. 25*(1), 85–104.

Embrechts, P., A. Höing, and A. Juri (2003). Using copulae to bound the Value-at-Risk for functions of dependent risks. *Finance Stoch. 7*(2), 145–167.

Embrechts, P. and G. Puccetti (2005). Bounds for functions of dependent risks. *Finance Stoch.* to appear

Embrechts, P. and G. Puccetti (2005). Aggregating risk capital, with an application to operational risk. *Preprint, ETH Zurich*

Li, H., M. Scarsini, and M. Shaked (1996). Bounds for the distribution of a multivariate sum. In *Distributions with Fixed Marginals and Related Topics*, Volume 28, pp. 198–212. Hayward, CA: Inst. Math. Statist.

Moscadelli, M. (2004). The modelling of operational risk: experience with the analysis of the data collected by the Basel Committee. Preprint, Banca d'Italia.

Ramachandran, D. and L. Rüschendorf (1995). A general duality theorem for marginal problems. *Probab. Theory Related Fields 101*(3), 311–319.

Rüschendorf, L. (1982). Random variables with maximum sums. *Adv. in Appl. Probab. 14*(3), 623–632.