Smile Asymptotics for Affine Stochastic Volatility Models

Martin Keller-Ressel

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TU Wien, Vienna

affine stochastic volatility models (ASVMs)

X_t ... log-price-process
V_t ... stochastic variance process

- (X_t, V_t) ist a Markov process
- The joint cumulant generating function is affine:

$$\Phi(t,u,w) := \log \mathbb{E}[\exp(uX_t + wV_t)] = \\ = \phi(t,u,w) + V_0\psi(t,u,w) + X_0 u$$

- There exists a set $\mathcal{U} \subset \mathbb{R}^2$, such that $\Phi(t,u,w) < \infty$ for all $(t,u,w) \in \mathbb{R}_{\geqslant 0} \times \mathcal{U}$ and $0 \in \mathcal{U}^\circ$.

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This result is crucial:

Generalized Riccati Equations

There exist functions $F, R : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\}$, such that:

$$\partial_t \phi(t,u,w) = F(u, \psi(t,u,w)), \quad \phi(0,u,w) = w$$
$$\partial_t \psi(t,u,w) = R(u, \psi(t,u,w)), \quad \psi(0,u,w) = w$$

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$R(u,w)$ ... state-dependent dynamics

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F, R are of Levy-Khintchine form

Mean-reversion function

The mean reversion function $\lambda(u)$ of V_t is given by

$$\lambda(u) := \frac{\partial R}{\partial w}(u, 0) .$$

Martingale Condition

$S_t = \exp(X_t)$ is a martingale if and only if $F(1, 0) = R(1, 0) = 0$ and there exists $\epsilon > 0$ such that

$$\int_0^{\epsilon} \frac{d\eta}{R(1, \eta)} = \infty .$$

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A simple sufficient condition for $(S_t)_{t \geq 0}$ to be a martingale is $F(1,0) = R(1,0) = 0$ and $\lambda(1) < \infty$.

Heston model

Heston in SDE form

$$
\begin{aligned}
dX_t &= -\frac{1}{2} V_t \, dt + \sqrt{V_t} \, dW_t^1 \\
dV_t &= -\lambda (V_t - \theta) \, dt + \gamma \sqrt{V_t} \, dW_t^2 \\
\left\langle dW_t^1, dW_t^2 \right\rangle &= \rho \, dt
\end{aligned}
$$

Heston in dual form

$$
\begin{aligned}
F(u, w) &= \lambda \theta w \\
R(u, w) &= -\frac{1}{2} u - \lambda w + \frac{u^2}{2} + \frac{\gamma^2 w^2}{2} + \rho \gamma w u
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Heston with added jumps ('Bates model')

Bates in SDE form

$$dX_t = \left(\mu - \frac{1}{2}V_t\right) dt + \sqrt{V_t}\, dW_t^1 + dJ_t$$

$$dV_t = -\lambda(V_t - \theta)\, dt + \gamma\sqrt{V_t}\, dW_t^2$$

Bates in dual form

$$F(u, w) = \lambda\theta w + \kappa(u) - u\kappa(1)$$

$$R(u, w) = -\frac{1}{2}u - \lambda w + \frac{u^2}{2} + \frac{\gamma^2 w^2}{2} + \rho\gamma wu$$

$\kappa(u) = \log \mathbb{E}[e^{J_1}]$ ... char. exponent von J_t.

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Bates 2000

Bates 2000 in SDE form

$$dX_t = \left(\mu - \frac{1}{2}\right) V_t \, dt + \sqrt{V_t} \, dW_t^1 + dJ_t(V_t)$$
$$dV_t = -\lambda(V_t - \theta) \, dt + \gamma\sqrt{V_t} \, dW_t^2$$

The intensity of $J_t(V_t)$ is *proportional* to V_t.

Bates 2000 in dual form

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Barndorff-Nielsen-Shephard (BNS)

BNS in SDE form

$$dX_t = (\mu - \frac{1}{2}V_t^2)dt + \sqrt{V_t}\,dW_t + \rho\,dJ_{\lambda t}$$
$$dV_t = -\lambda V_t\,dt + dJ_{\lambda t}$$

BNS in dual form

$$F(u,w) = \lambda\kappa(w+\rho u) - u\lambda\kappa(\rho)$$
$$R(u,w) = \frac{1}{2}(u^2 - u) - \lambda w$$

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Lemma

Suppose $(S_t)_{t\geq 0}$ is a martingale and the mean-reversion function $\lambda(u)$ satisfies

$$\lambda(0) < 0 \quad \text{and} \quad \lambda(1) < 0\,.$$

Then there exists a unique continuously differentiable function $w(u) : [0,1] \to \mathbb{R}_{\leqslant 0}$ such that

$$R(u, w(u)) = 0 \qquad \text{for all} \quad u \in [0,1]$$

and $w(0) = w(1) = 0$.

Long-Term Asymptotics of ϕ, ψ

$w(u)$ and $m(u) := F(u, w(u))$ are cumulant generating functions of infinitely divisible random variables and

$$\lim_{t\to\infty} \psi(t,u) = w(u)\ ,$$
$$\lim_{t\to\infty} \frac{1}{t}\phi(t,u) = m(u)$$

for all $u \in [0,1]$.

Write the price of a European call as Fourier Integral, and use a saddlepoint approximation:

$$\frac{1}{S_0}C(T,\xi) = 1 - \frac{e^{(1-u_*)\xi}}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-iz\xi}\exp\left(\Phi_T(u_* + iz)\right)}{(z + i(1-u_*))(z - iu_*)}\,dz =$$

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$$= 1 - \frac{\exp\left((1-u_*)\xi + Tm(u_*) + C\right)}{2u_*(1-u_*)\pi\sqrt{T}}\sqrt{\frac{1}{2\pi m''(u_*)}} +$$
$$+ \mathcal{O}\left(\frac{1}{T}\right)$$

under the condition $m'(u_*) = 0$.

Comparing with a Black-Scholes price yields the following result:

Long-term Asymptotics for the volatility smile

Let u_* be the solution of

$$m'(u_*) = 0 \; .$$

then

$$\sigma^2_{\text{imp}}(T,\xi)\Big|_{\xi=0} = -8m(u_*) + \mathcal{O}(T^{-1})$$
$$\frac{\partial}{\partial \xi}\sigma^2_{\text{imp}}(T,\xi)\Big|_{\xi=0} = \frac{1}{T}(8u_* - 4) + \mathcal{O}(T^{-2})$$

for $T \to \infty$.

- Approach has been used by Lewis for diffusion models.
- In the complex plane u_* is a saddlepoint of m.
- Problem: $m'(u_*) = 0$ can not be solved explicitly in most cases.
- Even if it can (e.g. Heston model), the resulting expression is complicated.
- Idea: Expand in some model parameter

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In the Heston model, expand in γ: (vol-of-vol parameter)

$$\begin{aligned} \text{ATMVariance} \quad &\sim \quad \theta + \frac{\theta\rho}{2\lambda}\gamma + \frac{\theta(5\rho^2 - 1)}{16\lambda^2}\gamma^2 + \mathcal{O}(\gamma^3) \\ T \cdot \text{ATMVarSkew} \quad &\sim \quad \frac{\rho}{\lambda}\gamma + \frac{\rho^2}{2\lambda^2}\gamma^2 + \mathcal{O}(\gamma^3) \end{aligned}$$

In the BNS model, expand in a scale parameter s of the jump distribution ($\kappa(u) = \kappa_0(su)$):

$$\mathrm{ATMVariance} \sim \kappa_0'(0)s + \frac{16\lambda^2\rho^2 + 8\lambda\rho - 1}{16\lambda}\kappa_0''(0)s^2 + (s^3)$$

$$T \cdot \mathrm{ATMVarSkew} \sim \rho\frac{\kappa_0''(0)}{\kappa_0'(0)}s + \mathcal{O}(s^2)$$

In the Bates 2000 model expand in γ **and** in a scale parameter s:

$$\begin{aligned} \mathrm{ATMVariance} &\sim \theta + \frac{\theta\rho}{2\lambda}\gamma + \theta\kappa_0''(0)s^2 + \mathcal{O}(\gamma^2 + s^2\gamma) \\ T \cdot \mathrm{ATMVarSkew} &\sim \frac{\rho}{\lambda}\gamma + \frac{\rho^2}{2\lambda^2}\gamma^2 + \mathcal{O}(\gamma^3 + \gamma^2 s^2) \end{aligned}$$

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Smile Asymptotics: $T \longrightarrow 0$

Medvedev & Scaillet consider a jump-diffusion Modell:

$$dX_t = \left(-\frac{\sigma_t^2}{2} - \mu_t(\sigma_t) \right) dt + \sigma_t dW_t^1 + dJ_t(\sigma_t)$$

$$d\sigma_t = a(\sigma_t)dt + b(\sigma_t)dW_t^2$$

$\rho \ldots$ Correlation of W_t^1, W_t^2.
$\lambda(\sigma_t) \ldots$ jump intensity of $J_t(\sigma_t)$

- Jumps can be added to the volatility process; do not influence short-term asymptotics.

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For $T \longrightarrow 0$ the following holds:

At-the-Money Vol

$$\sigma_{\text{imp}}\big|_{\xi=0} = \sigma_0 + \mathcal{O}(\sqrt{T})$$

At-the-Money Skew

$$\left.\frac{\partial \sigma_{\text{imp}}}{\partial \xi}\right|_{\xi=0} = \left[\frac{b\rho}{2\sigma_0} + \frac{\mu}{\sigma_0}\right] + \mathcal{O}(\sqrt{T})$$

μ ...jump compensator of J_t:

$$\mu = \lambda \int_{-\infty}^{\infty} (e^x - 1)\ \nu(dx)$$

Implication for the relationship of Skew vs. Volatility:

$$\mathrm{Skew} \propto \frac{a}{\mathrm{Vola}} \qquad \text{In Heston, BNS, Bates}$$

$$\mathrm{Skew} \propto \frac{a}{\mathrm{Vola}} + b \cdot \mathrm{Vola} \qquad \text{In Bates2000}$$

- Is the *b*-term needed?
- Crucial for the valuation of exotic options depending on forward skew, such as Cliquets with local caps and floors.

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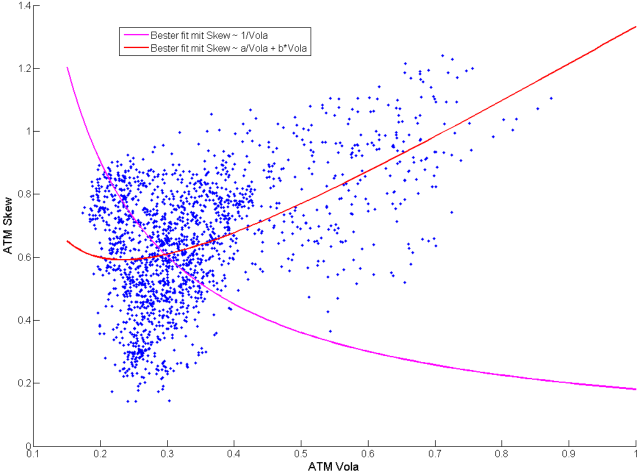
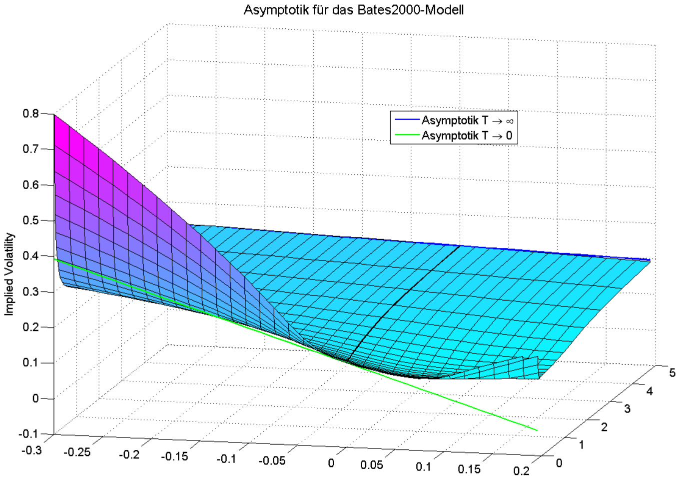

Bester fit mit Skew ~ 1/Vola
Bester fit mit Skew ~ a/Vola + b*Vola
ATM Skew
ATM Vola
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Asymptotik für das Bates2000-Modell
Asymptotik T → ∞
Asymptotik T → 0
Implied Volatility
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.3
-0.25
-0.2
-0.15
-0.1
-0.05
0
0.05
0.1
0.15
0.2
0
1
2
3
4
5

Lee's Moment Formula

Define

$$\omega_+(T) := \sup \{\omega : \mathbb{E}S_T^\omega < \infty\}$$
$$\omega_-(T) := \inf \{\omega : \mathbb{E}S_T^\omega < \infty\} \ .$$

Then

$$\limsup_{\xi \to \infty} \frac{\sigma_{\mathrm{imp}}^2(T,\xi)}{|\xi|} = \frac{\varsigma(\omega_+(T) - 1)}{T}$$
$$\limsup_{\xi \to -\infty} \frac{\sigma_{\mathrm{imp}}^2(T,\xi)}{|\xi|} = \frac{\varsigma(\omega_-(T))}{T}$$

where $\varsigma(x) = 2 - \left(4\sqrt{x^2 + x} - x\right)$.

- We call ω_+ upper critical moment and ω_- lower critical moment.
- From conservativeness and the martingale property it follows that $\omega_- \leq 0$ and $\omega_+ \geq 1$.
- The inverse function of $\omega_\pm(T)$ is $T_*(\omega)$, the time of moment explosion:

$$T_*(\omega) := \sup \{ T \geq 0 : \mathbb{E}[S_T^\omega] < \infty \} \qquad (\omega \in \mathbb{R} \setminus [0,1]) \,.$$

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Moment Explosions in ASVMs

Let $\omega \in \mathbb{R} \setminus [0,1]$ and suppose that $(S_t)_{t\geq 0}$ is a martingale and that $\lambda(\omega) \neq \pm\infty$. Define

$$\begin{aligned} f_+ &:= \sup\left\{\eta \geq 0 : F(\omega,\eta) < \infty\right\} \\ r_+ &:= \sup\left\{\eta \geq 0 : R(\omega,\eta) < \infty\right\} \\ r_0 &:= \inf\left\{\eta \geq 0 : R(\omega,\eta) = 0\right\} \ . \end{aligned}$$

- If $F(\omega,0) < \infty$, $R(\omega,0) < \infty$ and $r_0 < \min(f_+, r_+)$, then

$$T_*(\omega) = +\infty \ .$$

$\hookrightarrow$

Moment Explosions (ctd.)

- If $F(\omega, 0) < \infty$, $R(\omega, 0) < \infty$ and $r_0 \geq \min(f_+, r_+)$, then

$$T_*(\omega) = \int_0^{\min(f_+, r_+)} \frac{1}{R(\omega, \eta)}\, d\eta \,.$$

- If $F(\omega, 0) = \infty$ or $R(\omega, 0) = \infty$, then

$$T_*(\omega) = 0 \,.$$

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$T_*(\omega)$ can be inverted to give the upper critical moment $\omega_+(T)$ and the lower critical moment $\omega_-(T)$.

Moment Explosions in the Heston model

In the Heston model, the explosion time is given by

$$T_*(\omega) = \begin{cases} +\infty & b < 0, \Delta > 0 \\ \frac{1}{\sqrt{\Delta}} \log\left(\frac{b+\sqrt{\Delta}}{b-\sqrt{\Delta}}\right) & b > 0, \Delta > 0 \\ \frac{2}{\sqrt{-\Delta}}\left(\arctan\frac{\sqrt{-\Delta}}{b} + \pi \mathbf{1}_{\{b<0\}}\right) & \Delta < 0\ . \end{cases}$$

where $b = \rho\omega\gamma - \lambda$ and $\Delta = b^2 - \gamma^2\omega(\omega - 1)$.

Moment Explosions in the BNS model

Let $\kappa_+ := \sup\{u > 0 : \kappa(u) < \infty\}$. Then

$$T_* = \int_0^{f_+ \wedge r_+} \frac{d\eta}{R(\omega, \eta)} = -\frac{1}{\lambda} \log\left(1 - \frac{2\lambda(\kappa_+ - \rho\omega)}{\omega(\omega - 1)}\right) .$$

The critical moment functions $\omega_\pm(T)$ are given by

$$\omega_\pm(T) = \frac{1}{2} - \rho\frac{\lambda}{1 - e^{-\lambda T}} \pm \\ \pm \sqrt{\frac{1}{4} + (2\kappa_+ - \rho)\frac{\lambda}{1 - e^{-\lambda T}} + \rho^2 \frac{\lambda^2}{(1 - e^{-\lambda T})^2}} .$$

Forward Start Options

Today ($t = 0$) the seller and buyer agree on

- A start date τ.
- A time-to-maturity T.
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The payoff at time $\tau + T$ (for a call-type contract) is

$$\left(\frac{S_{\tau+T}}{S_\tau} - M \right)_+$$

For forward-start options we define moneyness ξ as $\xi = \log M + rT$.

- Several Exotic options have forward-start features built in.
- By comparing with BS-prices, we can for each τ define a 'forward smile' parameterized by (ξ, T).
- In an exp-Levy model: forward smile = spot smile
- In a local vol model the forward smile is flatter than the spot smile
- In an ASVM the forward smile is steeper than the spot smile and converges[1] to an 'asymptotic smile' as $\tau \to \infty$.

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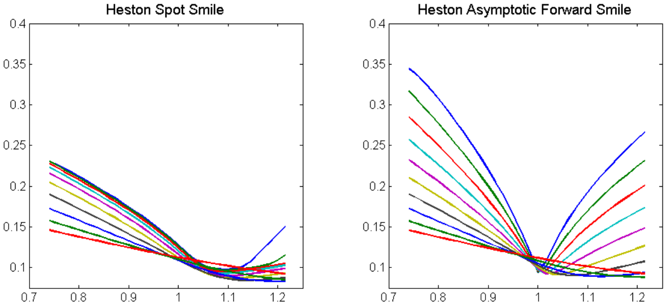
Heston Spot Smile
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.7
0.8
0.9
1
1.1
1.2
Heston Asymptotic Forward Smile
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.7
0.8
0.9
1
1.1
1.2

How to calculate the asymptotic forward smile?

Lemma

Suppose that $\lambda(0) < 0$ and that $F(u, w)$ satisfies a logarithmic moment condition. Then $(V_t)_{t \geq 0}$ converges in law to its stationary distribution with cumulant generating function

$$I(w) = -\int_0^w \frac{F(0,\eta)}{R(0,\eta)}\, d\eta \,.$$

The asymptotic forward smile

Let $C(\tau, T, \xi)$ be the price of a forward call.
Then

$$\lim_{\tau \to \infty} e^{r\tau} C(\tau, T, \xi) = \mathbb{E}\left[(e^{\widetilde{X}_T} - e^{\xi})_+\right] ,$$

where $\widetilde{X}_T$ is a random variable with cumulant generating function

$$\log \mathbb{E}[e^{\widetilde{X}_T u}] = \phi(T, u) + I(\psi(T, u)) .$$

Consider 'forward moments' $\mathbb{E}[e^{\widetilde{X}_T \omega}]$, and analyze the extreme-strike behaviour of the asymptotic forward smile as we did for the spot smile.

Moment Explosions in ASVMs

Let $\omega \in \mathbb{R} \setminus [0,1]$ and suppose that $(S_t)_{t \geq 0}$ is a martingale and that $\lambda(\omega) \neq \pm\infty$. Define

$$\begin{aligned} f_+ &:= \sup \{\eta \geq 0 : F(\omega, \eta) < \infty\} \\ r_+ &:= \sup \{\eta \geq 0 : R(\omega, \eta) < \infty\} \\ r_0 &:= \inf \{\eta \geq 0 : R(\omega, \eta) = 0\} \ . \end{aligned}$$

- If $F(\omega, 0) < \infty$, $R(\omega, 0) < \infty$ and $r_0 < \min(f_+, r_+)$, then

$$T_*^F(\omega) = +\infty \ .$$

$\hookrightarrow$

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Moment Explosions (ctd.)

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- If $F(\omega, 0) = \infty$ or $R(\omega, 0) = \infty$, then

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- In the Heston model $f_+ = r_+ = \infty$, but $l_+ = \frac{2\lambda}{\gamma}$.
- Instead of integrating from 0 to ∞, we now integrate only from 0 to $\frac{2\lambda}{\gamma}$.
- Idea: Explosion time getting smaller $\Leftrightarrow$ $|\omega_\pm^F|$ getting smaller $\Leftrightarrow$ Smile getting steeper.
- These statements can be quantified for the Heston and other ASVMs.

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