Extended Libor Models and Their Calibration

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Overview

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Forward Libor Models

- **Tenor structure:** $0 = T_0 < T_1 < \ldots < T_M < T_{M+1}$ with accrual periods $\delta_i := T_{i+1} - T_i$

- **Zero coupon bonds:** $B_k(t), \ t \in [0, T_k]$ with $B_k(T_k) = 1$

- **Forward Libor rates:** $L_1(t), \ldots, L_M(t)$

\[ L_k(t) = \frac{1}{\delta_k} \left( \frac{B_k(t)}{B_{k+1}(t)} - 1 \right), \quad t \in [0, T_k], \quad k = 1, \ldots, M \]

Remark

$L_1, \ldots, L_M$ are based on simple compounding that is an investor receives $1$ at $T_k$ and pays $1 + \delta_k L_k(t)$ at $T_{k+1}$
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Libor Model under $P_{M+1}$

For $i = 1, \ldots, M$

$$\frac{dL_i(t)}{L_i(t)} = A_i^{(M+1)}(dt) + \underbrace{\Gamma_i^T d\mathcal{W}^{(M+1)}(t)}_{\text{Continuous Part}} + \underbrace{\int_E \psi_i(t, u) \left( \mu - \nu^{(M+1)} \right)(dt, du)}_{\text{Jump Part}}$$

- $A_i$ are predictable drift processes
- $\mathcal{W}^{(M+1)}$ is a $D$-dimensional Brownian motion under $P_{M+1}$
- $\Gamma_i$ are predictable $D$-dimensional volatility processes
- $\omega \rightarrow \mu(dt, du, \omega)$ is a random point measure on $\mathbb{R}_+ \times E$ with $P_{M+1}$-compensator $\nu^{(M+1)}(dt, du)$
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Libor Model under $P_{M+1}$

We consider random point measures $\mu$ of finite activity satisfying

$$\int_E \psi_i(t, u) \left( \mu - \nu^{(M+1)} \right) (dt, du) = d \left[ \sum_{l=1}^{N_t} \psi_i(s_l, u_l) \right],$$

where $(s_l, u_l) \in \mathbb{R}_+ \times E$ are jumps of $\mu$ and

- $N_t$ is a Poisson process with intensity $\lambda$
- $E = \mathbb{R} \times \ldots \times \mathbb{R}$
- $u_l \in \mathbb{R}^m$ is distributed with $p_1(dx_1) \cdot \ldots \cdot p_m(dx_m)$
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Drift term under $P_{M+1}$

The requirement that $L_i$ is a martingale under $P^{(M+1)}$ implies

$$A_i^{(M+1)}(dt) = - \sum_{j=i+1}^{M} \frac{\delta_j L_j}{1 + \delta_j L_j} \Gamma_i^\top \Gamma_j dt + \lambda(t) dt \int_{\mathbb{R}^m} \psi_i(u, t) p(du) \left[ \prod_{j=i+1}^{M} \left( 1 + \frac{\delta_j L_j - \psi_i(t, u)}{1 + \delta_j L_j} \right) \right].$$
Dynamic of $L_i$ under $P_{i+1}$

Since $L_i$ is a martingale under $P_{i+1}$

$$\frac{dL_i}{L_i} = \Gamma_i^\top dW^{(i+1)} + \int_E \psi_i(t,u)(\mu - \nu^{(i+1)})(dt, du),$$

where

$$dW^{(i+1)} = -\sum_{j=i+1}^{M} \frac{\delta_j L_j^-}{1 + \delta_j L_j^-} \Gamma_i dt + dW^{(M+1)}$$

is a standard Brownian motion under $P_{i+1}$ and

$$\nu^{(i+1)}(dt, du) = \nu^{(M+1)}(dt, du) \left[ \prod_{j=i+1}^{M} \left( 1 + \frac{\delta_j L_j^- \psi_j(t,u)}{1 + \delta_j L_j^-} \right) \right].$$
Dynamic of $L_i$ under $P_{i+1}$

The logarithmic version reads as

$$d \ln(L_i) = A^{(i+1)}(dt) + \Gamma_i^T dW^{(i+1)} + d \left[ \sum_{i=1}^{N_t} \phi_i(s_l, u_l) \right]$$

with $\phi_i = \ln(1 + \psi_i)$ and

$$A^{(i+1)}(dt) = -\frac{1}{2} |\Gamma_i|^2 dt - \int_{\mathbb{R}^m} \psi_i(t, u) \nu^{(i+1)}(dt, du)$$

Observation

For $i < M$ the new compensator $\nu^{(i+1)}$ is not deterministic and $\ln(L_i)$ is generally not affine under $P_{i+1}$.
Dynamic of $L_i$ under $P_{i+1}$

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Observation

For $i < M$ the new compensator $\nu^{(i+1)}$ is not deterministic and $\ln(L_i)$ is generally not affine under $P_{i+1}$. 
Caplets

The price of $j$-th caplet at time zero is given by

$$C_j(K) = \delta_j B_{j+1}(0) E_{P_{j+1}} [(L_j(T_j) - K)^+]$$
Pricing and Calibration

Pricing Caplets under \( P_{i+1} \)

In terms of \textbf{log-forward moneyness} \( \nu = \ln(K/L_j(0)) \)

\[
C_j(\nu) := \delta_j B_{j+1}(0)L_j(0)E_{P_{j+1}}[(e^{X_j(t)} - e^{\nu})^+] ,
\]

with \( X_j(t) = \log(L_j(t)) - \log(L_j(0)) \).

Define

\[
\mathcal{O}_j(\nu) = E_{P_{j+1}}[(e^{X_j(t)} - e^{\nu})^+] - (1 - e^{\nu})^+ ,
\]

then

\[
F\{\mathcal{O}_j\}(z) := \int_{\mathbb{R}} \mathcal{O}_j(\nu) e^{ivz} \, dz = \frac{1 - \Phi_{P_{j+1}}(z - i; T_j)}{z(z - i)} .
\]
Pricing Caplets under $P_{i+1}$

In terms of log-forward moneyness $\nu = \ln(K/L_j(0))$

$$C_j(\nu) := \delta_j B_{j+1} L_j(0) E_{P_{j+1}} [(e^{X_j(t)} - e^\nu)^+] ,$$

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$$\mathcal{O}_j(\nu) = E_{P_{j+1}} [(e^{X_j(t)} - e^\nu)^+] - (1 - e^\nu)^+ ,$$

then

$$F\{\mathcal{O}_j\}(z) := \int_{\mathbb{R}} \mathcal{O}_j(\nu) e^{i\nu z} \, dz = \frac{1 - \Phi_{P_{j+1}}(z - i; T_j)}{z(z - i)} .$$
Characteristics Function of $L_M$ under $P_{M+1}$

Since

$$d \ln(L_M) = -\frac{1}{2} |\Gamma_i|^2 dt + \Gamma_i^T d\mathcal{N}^{(M+1)}(t) + d \left[ \sum_{l=1}^{N_t} \phi_i(s_l, u_l) \right]$$

and $N_t$, $\mathcal{N}^{(M+1)}$ and $u_l$ are mutually independent

$$\Phi_{P_{M+1}}(z; T) = \Phi_{C_{P_{M+1}}}(z; T) \Phi_{J_{P_{M+1}}}(z; T),$$

where $\Phi_{C_{P_{M+1}}}(z; T)$ ( $\Phi_{J_{P_{M+1}}}(z; T)$) is the c.f. of continuous (jump) part.
Specification Analysis: Continuous Part

For some predictable vector volatility process \((v_1(t), \ldots, v_d(t))\) define

\[
\Gamma_i = \begin{pmatrix}
\sqrt{1 - r_{SV}^2 \gamma_{i1}} \\
\sqrt{1 - r_{SV}^2 \gamma_{i2}} \\
\vdots \\
r_{SV} \beta_{i1} \sqrt{v_1(t)} \\
r_{SV} \beta_{id} \sqrt{v_d(t)}
\end{pmatrix}, \quad W^{(M+1)} = \begin{pmatrix}
W_1^{(M+1)} \\
W_2^{(M+1)} \\
\vdots \\
W_d^{(M+1)} \\
\overline{W}_1^{(M+1)} \\
\vdots \\
\overline{W}_d^{(M+1)}
\end{pmatrix}
\]

with mutually independent \(d\)-dimensional Brownian motions \(W^{(M+1)}\) and \(\overline{W}^{(M+1)}\).
Specification Analysis: Continuous Part

Let

$$\gamma_i(t) = c_i g(T_i - t)e_i, \quad e_i \in \mathbb{R}^d,$$

where

- $c_i > 0$ are loading factors
- $g_i(\cdot)$ is a scalar volatility function
- $e_i$ are unit vectors coming from the decomposition of the correlation matrix $\zeta$

$$\zeta_{ij} = e_i^\top e_j, \quad 1 \leq i, j \leq M,$$

be a deterministic volatility structure of the input Libor market model calibrated to ATM caps and ATM swaptions.
Define a new **time independent** volatility structure via

\[
\beta_i^T \beta_j = \frac{1}{\min(i, j)} \sum_{k=1}^{\min(i,j)} \frac{1}{T_k} \int_0^{T_k} \gamma_i^T(t) \gamma_j(t) \, dt.
\]

**Remark**

The covariance of the process \(\xi_i(t) := \int_0^t \Gamma_i^T(t) dW^{(M+1)}\) satisfies

\[
\text{cov}(\xi_i(t), \xi_j(t)) \approx \int_0^t \gamma_i^T(t) \gamma_j(t) \, ds
\]

and is approximately the same as in the input LMM.
Specification Analysis: Continuous Part

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\textbf{Remark}

The covariance of the process \( \xi_i(t) := \int_0^t \Gamma_i^T(t) d\mathcal{W}^{(M+1)} \) satisfies

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Specification Analysis: Continuous Part

Two possible specifications for the volatility process $\nu$

- **Stochastic Volatility Heston Model**

  \[
  d\nu_k = \kappa_k (1 - \nu_k) dt + \sigma_k \varrho_k \sqrt{\nu_k} dW_k^{(M+1)} + \sigma_k \sqrt{1 - \varrho_k^2} \nu_k dV_k^{(M+1)},
  \]

- **Stochastic Volatility BN Model**

  \[
  d\nu_k = \kappa_k \nu_k dt + \sigma_k \varrho_k dW_k^{(M+1)} + \sigma_k \sqrt{1 - \varrho_k^2} dV_k^{(M+1)}.
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Specification Analysis: Continuous Part

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Specification Analysis: Continuous Part

Two possible specifications for the volatility process $v$

- **Stochastic Volatility Heston Model**

\[
dv_k = \kappa_k (1 - v_k) dt + \sigma_k \rho_k \sqrt{v_k} dW_k^{(M+1)} + \sigma_k \sqrt{(1 - \rho_k^2)} \sqrt{v_k} dV_k^{(M+1)},
\]

- **Stochastic Volatility BN Model**

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dv_k = \kappa_k v_k dt + \sigma_k \rho_k dW_k^{(M+1)} + \sigma_k \sqrt{(1 - \rho_k^2)} dV_k^{(M+1)}.
\]
Specification Analysis

It holds

\[ \Phi_{P_{M+1}}^C(z; T) = \Phi_{D,P_{M+1}}^C(z; T) \Phi_{SV,P_{M+1}}^C(z; T), \]

where

\[ \Phi_{D,P_{M+1}}^C(z; T) = \exp \left( -\frac{1}{2} \theta^2_M(T) \left( z^2 + iz \right) \right), \quad \theta^2_M(T) = \int_0^T |\gamma_M|^2 \, dt \]

and

\[ \Phi_{SV,P_{M+1}}^C(z; T) = \exp \left( A_M(z; T) + B_M(z; T) \right) \]
Specification Analysis

In particular

\[ A_M(z; T) = \frac{\kappa_M}{\sigma_M^2} \left\{ (a_M + d_M) T - 2 \ln \left[ \frac{1 - g_M e^{d_M T}}{1 - g_M} \right] \right\} \]

\[ B_M(z; T) = \frac{(a_M + d_M)(1 - e^{d_M T})}{\sigma_M^2(1 - g_M e^{d_M T})} \]

and

\[ a_M = \kappa_M - i \rho_M \omega_M z \]
\[ d_M = \sqrt{a_M^2 + \omega_M^2(z^2 + i z)} \]
\[ g_M = \frac{a_M + d_M}{a_M - d_M}, \quad \omega_M = r_{SV} \beta_{MM} \sigma_M \]
Specification Analysis

As can be easily seen

\[
\lim_{z \to \infty} \frac{A_M(z; T)}{z} = -\alpha_M \omega_M \left( i \varrho_M + \sqrt{1 - \varrho_M^2} \right) T
\]

and

\[
\lim_{z \to \infty} \frac{B_M(z; T)}{z} = -\frac{\sqrt{1 - \varrho_M^2} + i \varrho_M}{\sigma_M}
\]

with

\[
\alpha_M := \frac{\kappa_M}{\sigma_M^2}
\]
Specification Analysis

Let us take $\phi_i(u, t) = u^\top \beta_i$, then the characteristic function of the jump part is given by

$$\Phi_{P_{M+1}}^J(z; T) = \exp \left( \lambda T \int_{\mathbb{R}} (e^{izv} - 1) \mu_M(v) \, dv \right),$$

where $\mu_M$ is the density of $u^\top \beta_M(t)$.

Observation

Due to the Riemann-Lebesgue theorem

$$\Phi_{P_{M+1}}^J(z; T) \to \exp(-\lambda T), \quad |z| \to \infty.$$
Specification Analysis

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*Due to the Riemann-Lebesgue theorem*

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$$
Specification Analysis: Asymptotic Properties

Computing sequentially

\[
\mathcal{L}_2 := \lim_{z \to \infty} \frac{\log(\Phi_{PM+1}(z; T))}{z^2},
\]

\[
\mathcal{L}_1 := \lim_{z \to \infty} \left[ \frac{\log(\Phi_{PM+1}(z; T))}{z} - (z + i)\mathcal{L}_2 \right],
\]

\[
\mathcal{L}_0 := \lim_{z \to \infty} \left[ \log(\Phi_{PM+1}(z; T)) - (z^2 + iz)\mathcal{L}_2 - z\mathcal{L}_1 \right],
\]

we get

\[
\mathcal{L}_0 = -\lambda, \quad \mathcal{L}_2 = -\frac{1}{2}\theta_M^2(T)
\]

and

\[
\text{Re} \mathcal{L}_1 = -\frac{\sqrt{1 - \varrho_M^2}}{\sigma_M} - \alpha_M\omega_M\sqrt{1 - \varrho_M^2} T, \quad \text{Im} \mathcal{L}_1 = -\frac{\varrho_M}{\sigma_M} - \alpha_M\omega_M\varrho_M T.
\]
Specification Analysis: Asymptotic Properties

Computing sequentially

\[ \mathcal{L}_2 := \lim_{z \to \infty} \log(\Phi_{PM_1}(z; T))/z^2, \]

\[ \mathcal{L}_1 := \lim_{z \to \infty} \left[ \log(\Phi_{PM_1}(z; T))/z - (z + i)\mathcal{L}_2 \right], \]

\[ \mathcal{L}_0 := \lim_{z \to \infty} \left[ \log(\Phi_{PM_1}(z; T)) - (z^2 + iz)\mathcal{L}_2 - z\mathcal{L}_1 \right], \]

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\[ \text{Re} \mathcal{L}_1 = -\sqrt{1 - \frac{\phi_M^2}{\sigma_M}} - \alpha_M \omega_M \sqrt{1 - \phi_M^2} T, \quad \text{Im} \mathcal{L}_1 = -\frac{\phi_M}{\sigma_M} - \alpha_M \omega_M \rho_M T \]
Parameters Estimation: Linearization

**Observation**

*From the knowledge of $L_1(T)$ for two different $T$ one can reconstruct all parameters of the SV process*

**Theorem**

$$
\psi_{PM+1}(z; T) := \log(\Phi_{PM+1}(z; T))
$$

$$
= L_2(z^2 + iz) + L_1 z + L_0 + R_0 + R_1(z),
$$

where $R_0 = R_0(\alpha_M, \kappa_M, \varphi_M, \omega_M)$ is a constant not depending on $\lambda$ and

$$
R_1(z) \to 0, \quad |z| \to \infty.
$$
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$$

where $R_0 = R_0(\alpha_M, \kappa_M, \varrho_M, \omega_M)$ is a constant not depending on $\lambda$ and

$$
R_1(z) \to 0, \quad |z| \to \infty.
$$
Parameters Estimation: Projection Estimators

We find estimates for $L_2$, $L_1$ and $L_0$ in the form of weighted averages

\[
\hat{L}_{2,U} := \int \text{Re}(\tilde{\Psi}_{P_{M+1}}(u)) w_2^U(u) \, du,
\]

\[
\hat{L}_{1,U} := \int \text{Im}(\tilde{\Psi}_{P_{M+1}}(u)) w_1^U(u) \, du - i\hat{L}_{2,U},
\]

\[
\hat{L}_{0,U} := \int \text{Re}(\tilde{\Psi}_{P_{M+1}}(u)) w_0^U(u) \, du - \hat{R}_0
\]

with

\[
\tilde{\Psi}_{P_{M+1}}(u) := \ln \left( 1 - u(u + i)F\{\tilde{O}_M\}(u + i) \right).
\]
Parameters Estimation

The weights are given by

\[ w_2^U = U^{-3} w_2(u/U), \quad w_1^U = U^{-2} w_1(u/U), \quad w_0^U = U^{-1} w_0(u/U), \]

where

\[ \int_{-1}^{1} w_2(u) du = 0, \quad \int_{-1}^{1} uw_2(u) du = 0, \quad \int_{-1}^{1} u^2 w_2(u) du = 1, \]

\[ \int_{-1}^{1} w_1(u) du = 0, \quad \int_{-1}^{1} uw_1(u) du = 1, \]

\[ \int_{-1}^{1} w_0(u) du = 1, \quad \int_{-1}^{1} uw_0(u) du = 0, \quad \int_{-1}^{1} u^2 w_0(u) du = 0. \]
Parameters Estimation: Jump distribution

Define

\[ F\{\hat{\mu}_M\}(z) = \tilde{\Psi}_{P_{M+1}}(z; T) - \hat{L}_2(z^2 + iz) - \hat{L}_1 z - \hat{L}_0 - \hat{R}_0 \]

or equivalently

\[ \hat{\mu}_M := F^{-1}\left[ \left( \tilde{\Psi}_{P_{M+1}}(\cdot; T) - \hat{L}_2(\cdot^2 + i\cdot) - \hat{L}_1 \cdot - \hat{L}_0 - \hat{R}_0 \right) 1_{[-U,U]}(\cdot) \right] \]

Remark

Due to lack of data and numerical errors \( \hat{\mu}_M \) may not be a density and needs to be corrected.
Parameters Estimation: Jump distribution

Define

\[ F\{\mu_M\}(z) = \tilde{\Psi}_{P_{M+1}}(z; T) - \mathcal{L}_2(z^2 + iz) - \mathcal{L}_1 z - \mathcal{L}_0 - \mathcal{R}_0 \]

or equivalently

\[ \hat{\mu}_M := F^{-1} \left[ \left( \tilde{\Psi}_{P_{M+1}}(\cdot; T) - \mathcal{L}_2(\cdot^2 + i\cdot) - \mathcal{L}_1 \cdot - \mathcal{L}_0 - \mathcal{R}_0 \right) 1_{[-U,U]}(\cdot) \right] \]

Remark

*Due to lack of data and numerical errors \( \hat{\mu}_M \) may not be a density and needs to be corrected.*
Parameters Estimation: Further optimization

Upon finding

\[
(\hat{\mathcal{L}}_0, U, \hat{\mathcal{L}}_1, U, \hat{\mathcal{L}}_2, U) \rightarrow \hat{T} := (\hat{\sigma}_M, \hat{\varrho}_M, \hat{\kappa}_M, \hat{\lambda})
\]

we may

- consider \( \hat{T} \) as a final set of parameters or
- consider nonlinear least-squares

\[
J(\mathcal{T}) = \sum_{i=1}^{N} w_i | C_M^T(K_i) - C_M(K_i) |^2
\]

and minimize \( J(\mathcal{T}) \) over the parametric set \( S \subset \mathbb{R}^4 \) taking as initial value \( \hat{T} \).
Approximative dynamics of \( L_i \) under \( P_{i+1} \)

It holds approximately

\[
\frac{dL_i}{L_i} \approx \Gamma_i^T dW^{(i+1)} + \int_E e^{u^T \beta_j} (\mu - \tilde{\nu}^{(i+1)})(dt, du),
\]

where \( dW^{(i+1)} \) is a standard Brownian motion under \( P_{i+1} \) and

\[
\tilde{\nu}^{(i+1)}(dt, du) = \nu^{(M+1)}(dt, du) \left[ \prod_{j=i+1}^M \left( 1 + \frac{\delta_j L_j(0) e^{u^T \beta_j}}{1 + \delta_j L_j(0)} \right) \right].
\]
Approximative dynamics of $v_k$ under $P_{i+1}$

By freezing the Libors at their initial values we obtain an approximative $v_k$ dynamics

$$
dv_k \approx \kappa_k^{(i+1)} \left( \theta_k^{(i+1)} - v_k \right) dt + \sigma_k \sqrt{v_k} \left( \varrho_k d\tilde{W}_k^{(i+1)} + \sqrt{1 - \varrho_k^2} d\tilde{W}_k^{(i+1)} \right)
$$

with reversion speed parameter

$$
\kappa_k^{(i+1)} := \left( \kappa_k - r_{SV} \sigma_k \varrho_k \sum_{j=i+1}^{M} \frac{\delta_j L_j(0)}{1 + \delta_j L_j(0)} \beta_{jk} \right),
$$

and mean reversion level

$$
\theta_k^{(i+1)} := \frac{\kappa_k^{(i+1)}}{\kappa_k^{(i+1)}}.
$$
Pricing Caplets under $P_{M+1}$

The price of $j$-th caplet at time zero can be alternatively written as

$$C_j(K) = \delta_j B_{M+1}(0) E_{P_{M+1}} \left[ \frac{B_{j+1}(T_j)}{B_{M+1}(T_j)} (L_j(T_j) - K)^+ \right]$$

Note that

$$\frac{B_{j+1}(T_j)}{B_{M+1}(T_j)} = \prod_{k=j+1}^{M} (1 + \delta_k L_k(T_j))$$

$$= \prod_{k=j+1}^{M} (1 + \delta_k) E_{\xi} \exp \left( \sum_{k=j+1}^{M} \xi_k \ln(L_k(T_j)) \right),$$

where $\{\xi_k\}_{k=j+1}^{M}$ are independent random variables and each $\xi_k$ takes two values 0 and 1 with probabilities $1/(1 + \delta_k)$ and $\delta_k/(1 + \delta_k)$. 
Pricing and Calibration

Pricing Caplets under $P_{M+1}$

Thus,

$$F\{O_j\}(z) = \frac{1 - E^\xi \Phi_{M+1}(z - i, \xi_{j+1}, \ldots, \xi_M)}{z(z - i)},$$

where $\Phi_{M+1}(z_j, z_{j+1}, \ldots, z_M)$ is the joint characteristic function of $(\ln(L_j(T_j)), \ldots, \ln(L_M(T_j)))$ under $P_{M+1}$.

Remark

Instead of terminal measure $P_{M+1}$ we could consider $P_{l+1}$ with $1 < l < M + 1$. 
Pricing and Calibration

Calibration Procedure

Pricing Caplets under $P_{M+1}$

Thus,

$$F\{\mathcal{O}_j\}(z) = \frac{1 - E_\xi \Phi_{M+1}(z - i, \xi_{j+1}, \ldots, \xi_M)}{z(z - i)},$$

where $\Phi_{M+1}(z, z_{j+1}, \ldots, z_M)$ is the joint characteristic function of $(\ln(L_j(T_j)), \ldots, \ln(L_M(T_j)))$ under $P_{M+1}$.

**Remark**

*Instead of terminal measure $P_{M+1}$ we could consider $P_{l+1}$ with $1 < l < M + 1$.***
Calibration results for 14.08.2007

Caplet volatilities for different caplet periods

- [17.5, 18]
- [15, 15.5]
- [12.5, 13]
- [10, 10.5]
Belomestny, D. and Spokoiny, V.
Spatial aggregation of local likelihood estimates with applications to classification,

Belomestny, D. and Reiβ, M.
*Spectral calibration of exponential Lévy models*,

Belomestny, D. and Schoenmakers, J.
A jump-diffusion Libor model and its robust calibration,
*SFB649 Discussion Paper*, 2006, **037**.