Extended Libor Models and Their Calibration

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Vienna, 16 November 2007

Overview

1. Introduction
   - Forward Libor Models
2. Modelling
   - Modelling under Terminal Measure
   - Modelling under Forward Measures
3. Pricing and Calibration
   - Pricing of Caplets
   - Specification Analysis
   - Calibration Procedure
4. Calibration in work

Forward Libor Models

- **Tenor structure:** $0 = T_0 < T_1 < \ldots < T_M < T_{M+1}$ with accrual periods $\delta_i := T_{i+1} - T_i$
- **Zero coupon bonds:** $B_k(t)$, $t \in [0, T_k]$ with $B_k(T_k) = 1$
- **Forward Libor rates:** $L_1(t), \ldots, L_M(t)$

$$L_k(t) = \frac{1}{\delta_k}\left(\frac{B_k(t)}{B_{k+1}(t)} - 1\right), \quad t \in [0, T_k], \quad k = 1, \ldots, M$$

Remark

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Libor Model under P_{M+1}

For $i = 1, \dots, M$

$$\frac{dL_i(t)}{L_i(t)} = A_i^{(M+1)}(dt) + \underbrace{\Gamma_i^\top d\mathcal{W}^{(M+1)}(t)}_{\text{Continuous Part}} + \underbrace{\int_E \psi_i(t,u)\left(\mu - \nu^{(M+1)}\right)(dt,du)}_{\text{Jump Part}}$$

- A_i are predictable drift processes
- $\mathcal{W}^{(M+1)}$ is a D-dimensional Brownian motion under P_{M+1}
- Γ_i are predictable D-dimensional volatility processes
- $\omega \to \mu(dt, du, \omega)$ is a random point measure on $\mathbb{R}_+ \times E$ with P_{M+1}-compensator $\nu^{(M+1)}(dt, du)$

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Libor Model under P_{M+1}

We consider random point measures μ of finite activity satisfying

$$\int_E \psi_i(t,u)\left(\mu - \nu^{(M+1)}\right)(dt,du) = d\left[\sum_{l=1}^{N_t} \psi_i(s_l,u_l)\right],$$

where $(s_l,u_l) \in \mathbb{R}_+ \times E$ are jumps of μ and

- N_t is a Poisson process with intensity λ
- $E = \underbrace{\mathbb{R} \times \ldots \times \mathbb{R}}_{m}$
- $u_l \in \mathbb{R}^m$ is distributed with $p_1(dx_1) \cdot \ldots \cdot p_m(dx_m)$

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Drift term under P_{M+1}

The requirement that L_i is a martingale under $P^{(M+1)}$ implies

$$A_i^{(M+1)}(dt) = -\sum_{j=i+1}^{M} \frac{\delta_j L_j}{1+\delta_j L_j} \Gamma_i^{\top} \Gamma_j dt + \\ + \lambda(t) dt \int_{\mathbb{R}^m} \psi_i(u,t) p(du) \left[\prod_{j=i+1}^{M} \left(1 + \frac{\delta_j L_{j-} \psi_i(t,u)}{1+\delta_j L_{j-}} \right) \right].$$

Dynamic of L_i under P_{i+1}

Since L_i is a martingale under P_{i+1}

$$\frac{dL_i}{L_i} = \Gamma_i^\top dW^{(i+1)} + \int_E \psi_i(t,u)(\mu - \nu^{(i+1)})(dt,du),$$

where

$$dW^{(i+1)} = -\sum_{j=i+1}^{M} \frac{\delta_j L_{j-}}{1+\delta_j L_{j-}} \Gamma_i dt + dW^{(M+1)}$$

is a standard Brownian motion under P_{i+1} and

$$\nu^{(i+1)}(dt,du) = \nu^{(M+1)}(dt,du) \left[\prod_{j=i+1}^{M} \left(1 + \frac{\delta_j L_{j-} \psi_j(t,u)}{1+\delta_j L_{j-}}\right) \right].$$

Dynamic of L_i under P_{i+1}

The logarithmic version reads as

$$d\ln(L_i) = A^{(i+1)}(dt) + \Gamma_i^\top dW^{(i+1)} + d\left[\sum_{i=1}^{N_t} \phi_i(s_l, u_l)\right]$$

with $\phi_i = \ln(1+\psi_i)$ and

$$A^{(i+1)}(dt) = -\frac{1}{2}|\Gamma_i|^2 dt - \int_{\mathbb{R}^m} \psi_i(t,u)\nu^{(i+1)}(dt,du)$$

Observation

For $i < M$ the new compensator $\nu^{(i+1)}$ is not deterministic and $\ln(L_i)$ is generally not affine under P_{i+1}.

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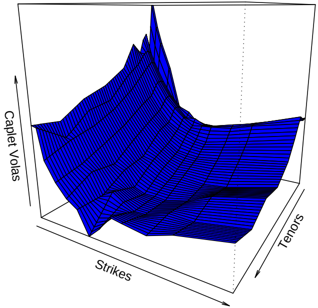
Caplets

The price of j-th caplet at time zero is given by

$$C_j(K) = \delta_j B_{j+1}(0) E_{P_{j+1}}[(L_j(T_j) - K)^+]$$



Pricing Caplets under P_{i+1}

In terms of **log-forward moneyness** $v = \ln(K/L_j(0))$

$$\mathcal{C}_j(v) := \delta_j B_{j+1}(0) L_j(0) E_{P_{j+1}}[(e^{X_j(t)} - e^v)^+],$$

with $X_j(t) = \log(L_j(t)) - \log(L_j(0))$.

Define

$$\mathcal{O}_j(v) = E_{P_{j+1}}[(e^{X_j(t)} - e^v)^+] - (1 - e^v)^+,$$

then

$$F\{\mathcal{O}_j\}(z) := \int_{\mathbb{R}} \mathcal{O}_j(v) e^{ivz}\, dz = \frac{1 - \Phi_{P_{j+1}}(z - i; T_j)}{z(z - i)}.$$

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Characteristic Function of L_M under P_{M+1}

Since

$$d\ln(L_M) = -\frac{1}{2}|\Gamma_i|^2\,dt + \Gamma_i^\top d\mathcal{W}^{(M+1)}(t) + d\left[\sum_{l=1}^{N_t}\phi_i(s_l,u_l)\right]$$

and N_t, $\mathcal{W}^{(M+1)}$ and u_l are mutually independent

$$\Phi_{P_{M+1}}(z;T) = \Phi^C_{P_{M+1}}(z;T)\Phi^J_{P_{M+1}}(z;T),$$

where $\Phi^C_{P_{M+1}}(z;T)$ ($\Phi^J_{P_{M+1}}(z;T)$) is the c.f. of continuous (jump) part.

Specification Analysis: Continuous Part

For some predictable vector volatility process $(v_1(t), \ldots, v_d(t))$ define

$$\Gamma_i = \begin{pmatrix} \sqrt{1 - r_{SV}^2}\gamma_{i1} \\ \sqrt{1 - r_{SV}^2}\gamma_{i2} \\ \vdots \\ \sqrt{1 - r_{SV}^2}\gamma_{id} \\ r_{SV}\beta_{i1}\sqrt{v_1(t)} \\ \vdots \\ r_{SV}\beta_{id}\sqrt{v_d(t)} \end{pmatrix}, \quad \mathcal{W}^{(M+1)} = \begin{pmatrix} W_1^{(M+1)} \\ W_2^{(M+1)} \\ \vdots \\ W_d^{(M+1)} \\ \overline{W}_1^{(M+1)} \\ \vdots \\ \overline{W}_d^{(M+1)} \end{pmatrix}$$

with mutually independent d-dimensional Brownian motions $W^{(M+1)}$ and $\overline{W}^{(M+1)}$.

Specification Analysis: Continuous Part

Let

$$\gamma_i(t) = c_i g(T_i - t) e_i, \quad e_i \in \mathbb{R}^d,$$

where

- $c_i > 0$ are loading factors
- $g_i(\cdot)$ is a scalar volatility function
- e_i are unit vectors coming from the decomposition of the correlation matrix ζ

$$\zeta_{ij} = e_i^\top e_j, \quad 1 \leq i, j \leq M,$$

be a deterministic volatility structure of the input **Libor market model** calibrated to ATM caps and ATM swaptions.

Specification Analysis: Continuous Part

Define a new **time independent** volatility structure via

$$\beta_i^\top \beta_j = \frac{1}{\min(i,j)} \sum_{k=1}^{\min(i,j)} \frac{1}{T_k} \int_0^{T_k} \gamma_i^\top(t)\gamma_j(t)\,dt.$$

Remark

The covariance of the process $\xi_i(t) := \int_0^t \Gamma_i^\top(t)d\mathcal{W}^{(M+1)}$ satisfies

$$\mathrm{cov}(\xi_i(t), \xi_j(t)) \approx \int_0^t \gamma_i^\top(t)\gamma_j(t)ds$$

and is approximately the same as in the input LMM.

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Specification Analysis: Continuous Part

Two possible specifications for the volatility process v

- **Stochastic Volatility Heston Model**

$$dv_k = \kappa_k(1-v_k)dt + \sigma_k \varrho_k \sqrt{v_k} d\overline{W}_k^{(M+1)} + \sigma_k \sqrt{(1-\varrho_k^2)} \sqrt{v_k} dV_k^{(M+1)},$$

- **Stochastic Volatility BN Model**

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Specification Analysis

It holds

$$\Phi^{C}_{P_{M+1}}(z;T) = \Phi^{C}_{D,P_{M+1}}(z;T)\,\Phi^{C}_{SV,P_{M+1}}(z;T),$$

where

$$\Phi^{C}_{D,P_{M+1}}(z;T) = \exp\left(-\frac{1}{2}\theta^2_M(T)\left(z^2 + \mathrm{i}z\right)\right), \quad \theta^2_M(T) = \int_0^T |\gamma_M|^2\,dt$$

and

$$\Phi^{C}_{SV,P_{M+1}}(z;T) = \exp\left(A_M(z;T) + B_M(z;T)\right)$$

Specification Analysis

In particular

$$A_M(z;T) = \frac{\kappa_M}{\sigma_M^2}\left\{(a_M + d_M)T - 2\ln\left[\frac{1 - g_M e^{d_M T}}{1 - g_M}\right]\right\}$$

$$B_M(z;T) = \frac{(a_M + d_M)(1 - e^{d_M T})}{\sigma_M^2(1 - g_M e^{d_M T})}$$

and

$$\begin{aligned} a_M &= \kappa_M - \mathrm{i}\varrho_M \omega_M z \\ d_M &= \sqrt{a_M^2 + \omega_M^2(z^2 + \mathrm{i}z)} \\ g_M &= \frac{a_M + d_M}{a_M - d_M}, \quad \omega_M = r_{SV}\beta_{MM}\sigma_M \end{aligned}$$

Specification Analysis

As can be easily seen

$$\lim_{z\to\infty}\frac{A_M(z;T)}{z} = -\alpha_M\omega_M\left(\mathrm{i}\varrho_M+\sqrt{1-\varrho_M^2}\right)T$$

and

$$\lim_{z\to\infty}\frac{B_M(z;T)}{z} = -\frac{\sqrt{1-\varrho_M^2}+\mathrm{i}\varrho_M}{\sigma_M}$$

with

$$\alpha_M := \frac{\kappa_M}{\sigma_M^2}$$

Specification Analysis

Let us take $\phi_i(u,t) = u^\top \beta_i$, then the characteristic function of the jump part is given by

$$\Phi^J_{P_{M+1}}(z;T) = \exp\left(\lambda T \int_{\mathbb{R}} (e^{izv} - 1)\mu_M(v)\,dv\right),$$

where μ_M is the density of $u^\top \beta_M(t)$.

Observation

Due to the Riemann-Lebesgue theorem

$$\Phi^J_{P_{M+1}}(z;T) \to \exp(-\lambda T), \quad |z| \to \infty.$$

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Specification Analysis: Asymptotic Properties

Computing sequentially

$$\mathcal{L}_2 := \lim_{z\to\infty} \log(\Phi_{P_{M+1}}(z;T))/z^2,$$

$$\mathcal{L}_1 := \lim_{z\to\infty} \left[\log(\Phi_{P_{M+1}}(z;T))/z - (z+\mathrm{i})\mathcal{L}_2\right],$$

$$\mathcal{L}_0 := \lim_{z\to\infty} \left[\log(\Phi_{P_{M+1}}(z;T)) - (z^2+\mathrm{i}z)\mathcal{L}_2 - z\mathcal{L}_1\right],$$

we get

$$\mathcal{L}_0 = -\lambda, \quad \mathcal{L}_2 = -\frac{1}{2}\theta_M^2(T)$$

and

$$\operatorname{Re}\mathcal{L}_1 = -\frac{\sqrt{1-\varrho_M^2}}{\sigma_M} - \alpha_M\omega_M\sqrt{1-\varrho_M^2}\,T, \quad \operatorname{Im}\mathcal{L}_1 = -\frac{\varrho_M}{\sigma_M} - \alpha_M\omega_M\varrho_M T$$

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Parameters Estimation: Linearization

Observation

From the knowledge of $\mathcal{L}_1(T)$ for two different T one can reconstruct all parameters of the SV process

Theorem

$$\Psi_{P_{M+1}}(z;T) := \log(\Phi_{P_{M+1}}(z;T)) = \mathcal{L}_2(z^2 + iz) + \mathcal{L}_1 z + \mathcal{L}_0 + R_0 + R_1(z),$$

where $R_0 = R_0(\alpha_M, \kappa_M, \varrho_M, \omega_M)$ is a constant not depending on λ and

$$R_1(z) \to 0, \quad |z| \to \infty.$$

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Parameters Estimation: Projection Estimators

We find estimates for $\mathcal{L}_2$, $\mathcal{L}_1$ and $\mathcal{L}_0$ in the form of weighted averages

$$
\begin{aligned}
\widehat{\mathcal{L}}_{2,U} &:= \int \mathrm{Re}(\widetilde{\Psi}_{P_{M+1}}(u))w_2^U(u)\,du, \\
\widehat{\mathcal{L}}_{1,U} &:= \int \mathrm{Im}(\widetilde{\Psi}_{P_{M+1}}(u))w_1^U(u)\,du - \mathrm{i}\widehat{\mathcal{L}}_{2,U}, \\
\widehat{\mathcal{L}}_{0,U} &:= \int \mathrm{Re}(\widetilde{\Psi}_{P_{M+1}}(u))w_0^U(u)\,du - \widehat{R}_0
\end{aligned}
$$

with

$$
\widetilde{\Psi}_{P_{M+1}}(u) := \ln\left(1 - u(u+\mathrm{i})F\{\widetilde{\mathcal{O}}_M\}(u+\mathrm{i})\right).
$$

Parameters Estimation

The weights are given by

$$w_2^U = U^{-3} w_2(u/U), \quad w_1^U = U^{-2} w_1(u/U), \quad w_0^U = U^{-1} w_0(u/U),$$

where

$$\int_{-1}^{1} w_2(u) du = 0, \int_{-1}^{1} u w_2(u) du = 0, \int_{-1}^{1} u^2 w_2(u) du = 1,$$

$$\int_{-1}^{1} w_1(u)\, du = 0, \int_{-1}^{1} u w_1(u)\, du = 1,$$

$$\int_{-1}^{1} w_0(u) du = 1, \int_{-1}^{1} u w_0(u) du = 0, \int_{-1}^{1} u^2 w_0(u) du = 0.$$

Parameters Estimation: Jump distribution

Define

$$F\{\widehat{\mu}_M\}(z) = \widetilde{\Psi}_{P_{M+1}}(z;T) - \widehat{\mathcal{L}}_2(z^2 + iz) - \widehat{\mathcal{L}}_1 z - \widehat{\mathcal{L}}_0 - \widehat{R}_0$$

or equivalently

$$\widehat{\mu}_M := F^{-1}\left[\left(\widetilde{\Psi}_{P_{M+1}}(\cdot;T) - \widehat{\mathcal{L}}_2(\cdot^2 + i\cdot) - \widehat{\mathcal{L}}_1 \cdot - \widehat{\mathcal{L}}_0 - \widehat{R}_0\right)\mathbf{1}_{[-U,U]}(\cdot)\right]$$

Remark

Due to lack of data and numerical errors $\widehat{\mu}_M$ may not be a density and needs to be corrected.

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Remark

Due to lack of data and numerical errors $\widehat{\mu}_M$ may not be a density and needs to be corrected.

Parameters Estimation: Further optimization

Upon finding

$$\left(\widehat{\mathcal{L}}_{0,U},\widehat{\mathcal{L}}_{1,U},\widehat{\mathcal{L}}_{2,U}\right)\longrightarrow\widehat{\mathcal{T}}:=\left(\widehat{\sigma}_M,\widehat{\varrho}_M,\widehat{\kappa}_M,\widehat{\lambda}\right)$$

we may

- consider $\widehat{\mathcal{T}}$ as a final set of parameters or
- consider nonlinear least-squares

$$\mathcal{J}(\mathcal{T})=\sum_{i=1}^{N}w_i\left|C_M^{\mathcal{T}}(K_i)-C_M(K_i)\right|^2$$

and minimize $\mathcal{J}(\mathcal{T})$ over the parametric set $\mathcal{S}\subset\mathbb{R}^4$ taking as initial value $\widehat{\mathcal{T}}$.

Approximative dynamics of L_i under P_{i+1}

It holds approximately

$$\frac{dL_i}{L_i} \approx \Gamma_i^\top dW^{(i+1)} + \int_E e^{u^\top \beta_i}(\mu - \tilde{\nu}^{(i+1)})(dt, du),$$

where $dW^{(i+1)}$ is a standard Brownian motion under P_{i+1} and

$$\tilde{\nu}^{(i+1)}(dt, du) = \nu^{(M+1)}(dt, du) \left[\prod_{j=i+1}^{M} \left(1 + \frac{\delta_j L_{j-}(0) e^{u^\top \beta_j}}{1 + \delta_j L_{j-}(0)} \right) \right].$$

Approximative dynamics of v_k under P_{i+1}

By freezing the Libors at their initial values we obtain an approximative v_k dynamics

$$dv_k \approx \kappa_k^{(i+1)} \left(\theta_k^{(i+1)} - v_k\right) dt + \sigma_k \sqrt{v_k} \left(\varrho_k d\widetilde{W}_k^{(i+1)} + \sqrt{1-\varrho_k^2} d\overline{W}_k^{(i+1)}\right)$$

with reversion speed parameter

$$\kappa_k^{(i+1)} := \left(\kappa_k - r_{SV} \sigma_k \varrho_k \sum_{j=i+1}^{M} \frac{\delta_j L_j(0)}{1+\delta_j L_j(0)} \beta_{jk}\right),$$

and mean reversion level

$$\theta_k^{(i+1)} := \frac{\kappa_k}{\kappa_k^{(i+1)}}.$$

Pricing Caplets under P_{M+1}

The price of j-th caplet at time zero can be alternatively written as

$$C_j(K) = \delta_j B_{M+1}(0) E_{P_{M+1}} \left[\frac{B_{j+1}(T_j)}{B_{M+1}(T_j)} (L_j(T_j) - K)^+ \right]$$

Note that

$$\frac{B_{j+1}(T_j)}{B_{M+1}(T_j)} = \prod_{k=j+1}^{M} (1 + \delta_k L_k(T_j))$$

$$= \prod_{k=j+1}^{M} (1 + \delta_k) E_\xi \exp\left(\sum_{k=j+1}^{M} \xi_k \ln(L_k(T_j)) \right),$$

where $\{\xi_k\}_{k=j+1}^{M}$ are independent random variables and each ξ_k takes two values 0 and 1 with probabilities $1/(1 + \delta_k)$ and $\delta_k/(1 + \delta_k)$.

Pricing Caplets under P_{M+1}

Thus,

$$F\{\mathcal{O}_j\}(z) = \frac{1 - E_\xi \Phi_{M+1}(z - \mathrm{i}, \xi_{j+1}, \ldots, \xi_M)}{z(z - \mathrm{i})},$$

where $\Phi_{M+1}(z_j, z_{j+1}, \ldots, z_M)$ is the joint characteristic function of $(\ln(L_j(T_j)), \ldots, \ln(L_M(T_j)))$ under P_{M+1}.

Remark

Instead of terminal measure P_{M+1} we could consider P_{l+1} with $1 < l < M + 1$.

Pricing Caplets under P_{M+1}

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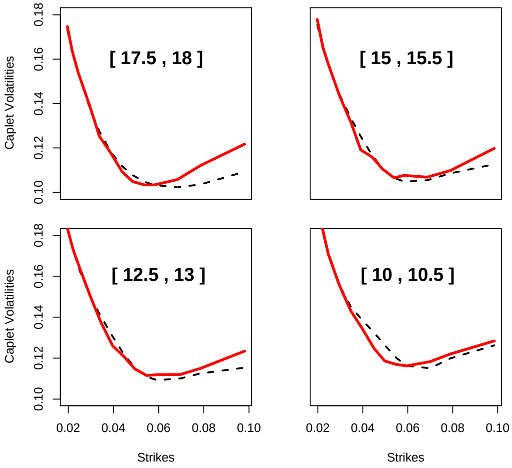
Remark

Instead of terminal measure P_{M+1} we could consider P_{l+1} with $1 < l < M + 1$.

Calibration results for 14.08.2007

Caplet volas for different caplet periods



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