Variance of hedging strategies in discrete time

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Joint work with Stefano Herzel

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Introduction

- Most mathematical models for financial markets assume continuous-time trading
- Strategies that are optimal in continuous time (e.g. BS delta hedging) need not to be optimal in discrete time
- Figlewski (1989)-*It is apparent that, simply by rebalancing discretely instead of continuously, we have departed markedly from the theoretical world of Black-Scholes*
- Can we effectively compute a discrete time optimal strategy?
- Can we measure the error due to time discretization?

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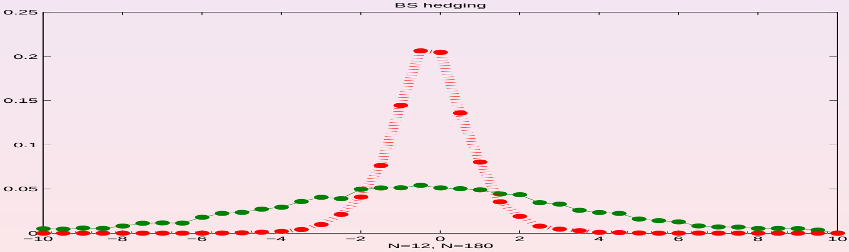
The discretization error in the BS model



The problem (Schweizer 1995)

- Hedging a European claim H with a self-financing strategy in a risky asset S and a riskless asset. We assume (wlog) zero interest rate.
- Let $S = (S_k)_{k=0}^N$ be the asset price process and $\vartheta = (\vartheta_k)_{k=1}^N$ be the hedging ratio of a self-financing trading strategy.
- The value of the hedging portfolio is

$$V_k = \vartheta_k S_k + \eta_k$$

- The cumulative gain process is

$$G_N(\vartheta) = \sum_{k=1}^{N} \vartheta_k \Delta S_k$$

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The problem (Schweizer 1995)

The basic problem:

$$\min_{\vartheta \in \Theta} E\left[\left(H - c - G_N(\vartheta) \right)^2 \right]$$

for fixed $c \in \mathbb{R}$

The existence of the optimal strategy

- Assume that process X satisfies the following Non-Degeneracy (ND) condition:

$$\frac{(E_{k-1}\Delta S_k)^2}{\text{var}_{k-1}\Delta S_k} < M$$

for all ω and k.

- Then there exists a unique optimal trading strategy ϑ^c that solves the basic problem (Th. 2.2)

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- A counterexample shows that the ND condition is necessary for the existence of a solution
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Variance of a sub-optimal strategy

- Starting from a value c, we follow a s.f. strategy to hedge the payoff H of a contingent claim. The hedging error of the strategy is

$$\varepsilon(\vartheta, c) = H - c - G_N(\vartheta).$$

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- A natural example is when $\vartheta = \Delta^N$ the BS-delta and S is the BS-process.
- Hayashi and Mykland (2005) showed that

$$\varepsilon(\Delta^N, c) - \sqrt{\frac{T}{2N}} \int_0^T \Gamma_u \sigma^2 S_u^2 dW_u^* \to 0$$

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- Toft (1996) provides an asymptotic approximation for the variance

$$\frac{1}{2}\sigma^4\left(\frac{T}{N}\right)^2\sum_{k=0}^{N-1}E_0\left[(\Gamma_k S_k^2)^2\right]$$

- Kamal and Derman (1999) propose a more *trader-friendly* approximation

$$\frac{\pi}{4N}\sigma^2 Vega^2$$

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The model framework

- Same setting as Hubalek, Kallsen and Krawczyk (2006)
- Let $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n \in (0,1,\ldots,N)}, P)$ be a filtered probability space. The risky asset is

$$S_n = S_0 \exp(X_n),$$

where the process $X = (X_n)$ for $n = 0, 1, \ldots, N$, satisfies

1. X is adapted to the filtration $\mathcal{F}_n, n \in (0, 1, \ldots, N)$,
2. $X_0 = 0$,
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- Can compute the moment generating function of ΔX

$$m(z) = E[e^{z\Delta X}],$$

assuming that it exists for $0 \leq Re(z) \leq 2$

- A rather general class of models, including Black-Scholes, Merton jump-diffusion, NIG, etc.

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The main idea

- Start by considering an *exponential claim*, that is a contingent claim with payoff

$$H(z) = S_N^z = S_0^z e^{zX_N}$$

- The i.i.d. assumption for ΔX_n makes computations easy for exponential claims.
- Next we consider a contingent claim which is a "linear combination" of exponential claims (let us call it *simple claim*)
- Use linearity properties and Fubini

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Simple claims

- A contingent claim is "simple" if its payoff is a "linear combination" of exponential claims:

$$H = \int S_N^z \Pi(dz)$$

The payoff of a simple claim is an inverse Laplace Transform

- A European call is a simple claim

$$(S_N - K)^+ = \frac{1}{2\pi i} \int_{R-i\infty}^{R+i\infty} S_N^z \frac{K^{1-z}}{z(z-1)} dz,$$

with $R > 1$ arbitrary

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Important observation

- The price at time t_{n-1} of a simple claim is

$$\begin{aligned} P_{n-1} &= E^Q_{n-1}\left[\int S^z_N \Pi(dz)\right] \\ &= \int E^Q_{n-1}[S^z_N]\Pi(dz) = \int S^z_{n-1} m_0(z)^{N-n+1}\Pi(dz), \end{aligned}$$

  where E^Q_{n-1} is the pricing measure and $m_0(z)$ is the m.g.f. of corresponding ΔX.

- The Delta of a simple claim is an inverse Laplace transform

$$\Delta_n = \int f(z)_n S^{z-1}_{n-1}\Pi(dz),$$

  where $f(z)_n = zm_0(z)^{N-n+1}$ does not depend on S_{n-1}.

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Delta hedging error

- The hedging error of a simple claim is an inverse Laplace transform

$$H - c - G_N(\Delta) = \int (H(z) - c - G_N(\Delta(z)))\Pi(dz)$$

- I can compute expected value and variance of the hedging error if I am able to compute things inside the integral

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Expected Delta hedging error

- $$E[H] = \int E[S_N^z]\Pi(dz) = \int S_0^z m(z)^N \Pi(dz)$$

- $$\begin{aligned} E[\Delta_n \Delta S_n] &= \int E[f(z)_n S_{n-1}^{z-1} \Delta S_n]\Pi(dz) \\ &= \int S_0^z f(z)_n (m(1)-1) m(z)^{n-1} \Pi(dz), \end{aligned}$$

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Variance of Delta hedging

- Need to compute all the covariances. It boils down to compute
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Main results

- Our results apply to (any linear combination of) simple claims and any ϑ given by

$$\vartheta_n = \int f(z)_n S_{n-1}^{z-1} \Pi(dz),$$

Examples: BS-Delta, Wilmott "improved delta", local optimal (already in Černý (2007))

- $$E[\varepsilon(\vartheta,0)] = \int S_0^z \left[m(z)^N - (m(1)-1) \sum_{k=1}^{N} f(z)_k m(z)^{k-1} \right] \Pi(dz)$$

$$E[\varepsilon(\vartheta,0)^2] = \int\int S_0^{y+z} V(y,z) \Pi(dz) \Pi(dy),$$

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- The formulas we wish to evaluate involve one- and two-dimensional Laplace transforms. There are at least two methods
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Inversion of Laplace transform

- Algorithms implemented in MATLAB
- One-dimensional case: we used "invlap.m" constructed by Hollenbeck (1998), very accurate
- Bi-dimensional case: we wrote a code based on Choudhury, Lucantoni, Whitt (1994), quite accurate

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Inversion of bi-dim Laplace transform

Absolute error (multiplied by 10^4) of variance of the hedging error of the delta strategy computed with the two-dimensional inversion algorithm for an at-the money call option with number of trading dates $N = 1$ as a function of m and n (truncation error). The other parameters used by the algorithm are $A_1 = A_2 = 30$ (aliasing error), $l_1 = l_2 = 1$ (roundoff error).

Inversion of bi-dim Laplace transform

m	n	err
20	40	494.21
50	50	156.21
50	100	33.91
100	100	20.01
100	200	4.31
200	200	2.51
20	400	1.01
50	400	0.91
100	400	0.71
20	600	0.31

Transaction Costs

- We consider the case of proportional transaction costs, i.e. Bid price $S^b = S(1 - k/2)$; Ask price $S^a = S(1 + k/2)$
- Transaction cost at time t_n

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\begin{aligned}
\mathrm{TC}_n &= |\vartheta_{n+1} - \vartheta_n| \frac{kS_n}{2} \\
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- The problem is to compute mean and variance of

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- We now assume that ϑ_{n+1} is an increasing function of S_n (the case of a decreasing function is analogous)
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$$\vartheta_{n+1}>\vartheta_n \Longleftrightarrow S_n>S_{n-1} \Longleftrightarrow \Delta X_n>0$$

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E\left[\mathbf{1}_{\Delta X_n>0}\vartheta_{n+1}S_n\right] &= E\left[\mathbf{1}_{\Delta X_n>0}S_n\int f(z)_{n+1}S_n^{z-1}\Pi(dz)\right] \\
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- With the same technique one may in principle compute moments of higher order
- Can choose independently a strategy ("$f(z)_n$") and a model ("$m(z)$")
- In particular, one can measure the hedging error of the Delta strategy for different data generating processes
- The result can be easily extended to the case when ΔX_n are not identically distributed

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Applications

- Assess precision of existing approximations of the variance
- How worse than optimal-variance is Delta hedging?
- Compare performances of various strategies for different models
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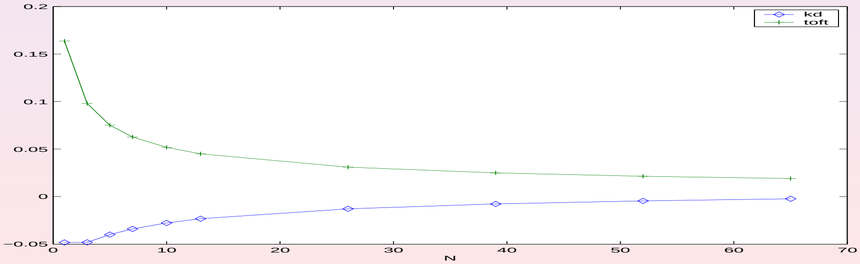
Relative errors of approximations of standard deviation



Model risk

- Data generating process: Merton jump-diffusion process with normally distributed jumps, with returns with annual mean $\mu \approx 0.14$ and volatility $\sigma \approx 0.44$
- Hedging ratios: B-S Delta and B-S locally optimal computed with parameters μ and σ, Merton optimal
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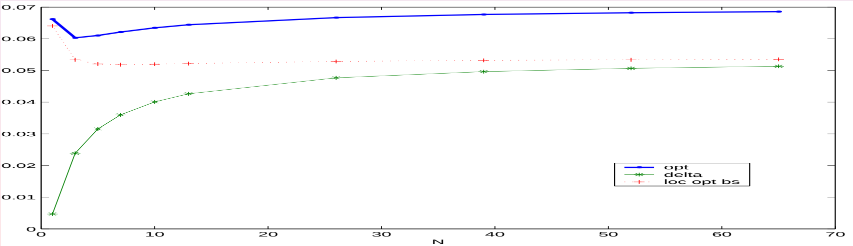
Model risk

Sharpe index of different strategies



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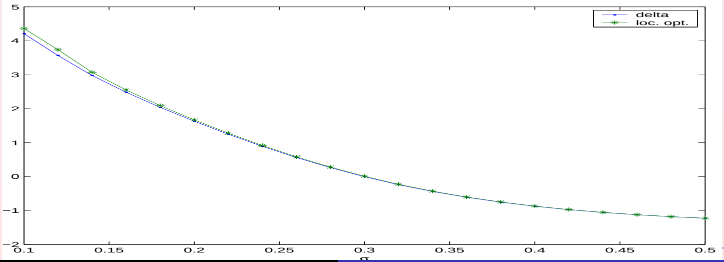
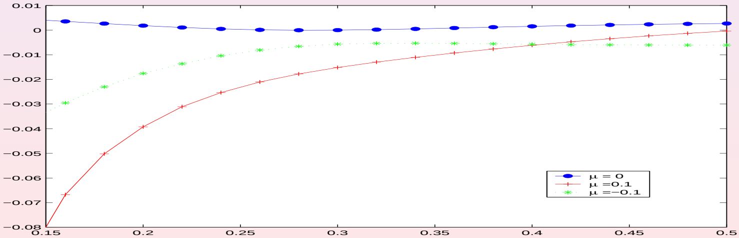
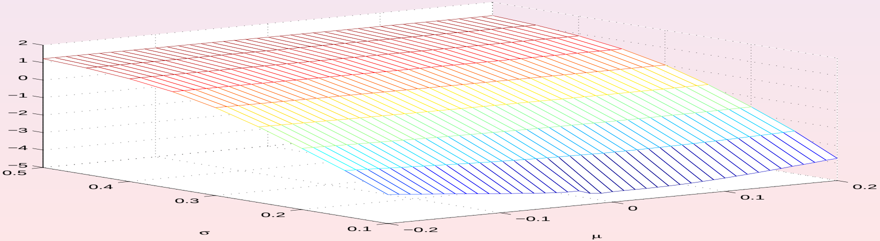
Sharpe index as a function of realized volatility σ, with $\mu_0 = \mu = 0.1$ and $N = 10$



$s(\Delta) - s(\xi^{lo})$ as a function of σ, for different μ ($\sigma_0 = 0.3$, $\mu_0 = 0.1$)



Sharpe index of local optimal strategy as a function of σ and μ ($\sigma_0 = 0.3$, $\mu_0 = 0$, $S = K = 100$, $N = 10$)



Conclusion

- We have an efficient way to compute moments of hedging errors of strategies for simple claims and for a wide class of data generating process
- This allows us to measure the performance of hedging strategies in different settings, for instance under model mispecification
- With the same approach we can also measure the influence of transaction costs
- Interesting topics to be explored:
  1. More general processes (GARCH)
  2. Multi-dimensional (delta-gamma hedging, basket options)
  3. Dependence on parameter estimates, robustness...

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