# Optimal dividend payments problem under time of ruin constraints: Exponential case (Talk)

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The idea of this work is to study a way to link the two standard problems in the optimal dividend payment theory: the maximization of profits and the minimization of the probability of ruin.

The ideal set up of this idea will be:

$$\sup_{D\in\Theta} \quad V^D(x) \quad \text{s.t.} \quad X_t^D = x + ct - \sum_{i=1}^{N_t} Y_i - D_t \quad \textit{and}, \ \varphi_D(x) \leq \epsilon$$

Where $V^D(x) := \mathbb{E}\Big[\int_0^{\tau^D} e^{-\delta t} dD_t\Big]$ represents the typical value function of the profit maximization problem when the dividend strategy $D_t$ is followed, $D_t$ being a non-negative, non-decreasing and adapted cadlag process. $\tau^D$ is the time of ruin, i.e. $\tau^D = \inf\{t \geq 0 : X_t^D < 0\}$. $X_t^D$ models the accumulation process, which follows the classical Cramer-Lundberg equation, when strategy $D_t$ is followed. $D_t$ represents the cumulative dividend process. $N_t$ represents the claim arrival process which is assumed to follow a Poisson process with intensity $\lambda$ and claims $Y_i$ iid with distribution $G(y)$. Finally, $\varphi_D(x)$ would be the probability of ruin function when strategy $D_t$ is followed, i.e. $\varphi_D(x) := Pr\{\tau^D < \infty\}$. However, this problem remains open.

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In this paper we study the following problem:

$$(P1) \qquad V(x) := \sup_{D\in\Theta} \quad V^D(x) \quad \text{s.t.} \quad X_t^D = x + ct - \sum_{i=1}^{N_t} Y_i - D_t$$

$$\mathbb{E}\left[\int_0^{\tau^D} e^{-\delta s} ds\right] \geq \int_0^T e^{-\delta s} ds \quad T \text{ fixed},$$

where all the variables are as above. However, we assume the claim sizes $\{Y_i\}$ have exponential distribution with intensity $\alpha$. The motivation behind such a constrain is that it imposes a restriction on the time of ruin.

In order to do so, we use Duality Theory. Hence, we can use Lagrange multipliers to obtain the following *Lagrange Dual* function and our problem becomes:

$$(P2) \qquad \mathcal{V}_\Lambda(x) := \sup_{D\in\Theta} \quad \mathbb{E}\left[\int_0^{\tau^D} e^{-\delta t} dD_t + \Lambda \int_0^{\tau^D} e^{-\delta t} dt\right] - \Lambda \int_0^T e^{-\delta s} ds$$

Since the last term does not depend on $D$ and is linear on $\Lambda$ we will first focus on the first term on the right hand side of this equation. For this term, it is known [1] that its solution $\mathcal{V}_\Lambda(x)$ must satisfy the following HJB equation

$$(1) \qquad max\{\Lambda + cV'(x) + \lambda \int_0^x V(x-y)dG(y) - (\lambda+\delta)V(x), 1 - V'(x)\} = 0,$$

and to have a barrier strategy $b^*$ as optimal strategy. We deduce that the optimal value of $b^*$ can be derived from the equation

$$(2) \quad (r_2 - r_1)(\alpha + r_1)(\alpha + r_2)\Lambda = -r_1 e^{-r_2 b}(r_1(\lambda+\delta) + \alpha\delta) + r_2 e^{-r_1 b}(r_2(\lambda+\delta) + \alpha\delta),$$

where $r_1$ and $r_2$ depend on the parameters of the model. Note that $\Lambda = 0$ leads to

$$r_1 e^{r_1 b}(r_1(\lambda+\delta) + \alpha\delta)) = r_2 e^{r_2 b}(r_2(\lambda+\delta) + \alpha\delta))$$

which is consistent with the standard problem without constraint [2]. The following proposition about the barrier level can be proved.

**Proposition 1.** *Assume $c, \alpha, \delta$ are not zero. Equation (2) has a unique real solution for $b$.*

We also prove the following:

**Theorem 2.** *The value function of $(P2)$ is given by: if $\alpha\lambda(c+\Lambda) - (\delta+\lambda)^2 \leq 0$*

$$\mathcal{V}_\Lambda(x) = x + \frac{c+\Lambda}{\lambda+\delta} + \frac{\Lambda}{\delta} e^{-\delta T}$$

*if $\alpha\lambda(c+\Lambda) - (\delta+\lambda)^2 > 0$*

$$(3) \qquad \mathcal{V}_\Lambda(x) = \begin{cases} \frac{\alpha c - \lambda - \delta}{\alpha\delta} + x - b + \frac{\Lambda}{\delta} e^{-\delta T} & if \quad x \geq b^* \\ C_1 e^{r_1 x} + C_2 e^{r_2 x} + \frac{\Lambda}{\delta} e^{-\delta T} & if \quad x \leq b^* \end{cases}$$

In order to find out the solution to $(P1)$ we have the following proposition.

**Proposition 3.** *For each* $x \geq 0$ *there exists* $(\Lambda^*, D^*)$*,* $\Lambda^* \geq 0$ *and* $D^*$ *the optimal strategy for* $(P2)$ *with* $\Lambda^*$*, such that* $\mathbb{E}[\int_0^{\tau^{D^*}} e^{-\delta s} ds] = \int_0^T e^{-\delta s} ds$*.*

As a consequence we have the main theorem:

**Theorem 4.** *Let* $V(x)$ *and* $\mathcal{V}_\Lambda(x)$ *be the optimal solution to* $(P1)$ *and* $(P2)$*, respectively, then:* $\inf\limits_{\Lambda \geq 0} \mathcal{V}_\Lambda(x) = V(x)$*.*

# References


[1] Thonhauser, Stefan. Albrecher, Hansjorg. 2007. *Dividend maximization under consideration of the time value of ruin.* Insurance: Mathematics and Economics 41 (2007). Pag: 163 - 184.

[2] Schmidli, Hanspeter. 2008. *Stochastic Control in Insurance.* Probability and Its Applications. Applied Probability Trust. Springer.