CONTAGION EFFECTS AND COLLATERALIZED CREDIT VALUE ADJUSTMENTS FOR CREDIT DEFAULT SWAPS

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The paper is concerned with counterparty credit risk for credit default swaps in the presence of default contagion. In particular, we study the impact of default contagion on credit value adjustments such as the BCCVA (Bilateral Collateralized Credit Value Adjustment) of Brigo et al. (2014) and on the performance of various collateralization strategies. We use the incomplete-information model of Frey and Schmidt (2012) for our analysis. We find that contagion effects have a substantial impact on the effectiveness of popular collateralization strategies. We go on and derive improved collateralization strategies that account for contagion. Theoretical results are complemented by a simulation study.

Keywords: Counterparty credit risk; bilateral credit value adjustment; collateralization; credit default swap; contagion; incomplete information.

1. Introduction

The distress of many financial firms in recent years has made counterparty risk management for over-the-counter (OTC) derivatives such as credit default swaps (CDS) an issue of high concern. Crucial tasks in this context are the computation of *credit value adjustments*, which account for the possibility that one of the contracting parties defaults before the maturity of the OTC contract, and the mitigation of counterparty risk by *collateralization*. Collateralization refers to the practice of posting securities (the so-called collateral) that serve as a pledge for the collateral taker. These securities are liquidated if one of the contracting parties defaults before maturity, and the proceeds are used to cover the replacement cost of the contract. In order to ensure that the funds generated in this way are sufficient, the collateral position needs to be adjusted dynamically in reaction to changes in the value of the underlying derivative security. The price dynamics of the collateral thus play a crucial role for the performance of a given collateralization strategy.

In the present paper we study the impact of different price dynamics on the size of value adjustments and on the performance of collateralization strategies for CDSs. We are particularly interested in the influence of contagion. Contagion effects - the fact that the default of a firm leads to a sudden increase in the credit spread of surviving firms - are frequently observed in financial markets; a prime example are the events that surrounded the default of Lehman Brothers in 2008. To see that contagion might be relevant for the performance of collateralization strategies consider the scenario where the protection seller defaults during the runtime of the CDS. In such a case contagion might lead to a substantial increase in the credit spread of the reference entity (the firm on which the CDS is written) and hence in turn to a much higher replacement value for the CDS. In standard collateralization strategies this is taken into account at most in a very crude way, and the amount of collateral posted before the default might be insufficient for replacing the CDS. In our view this issue merits a detailed analysis in the context of dynamic portfolio credit risk models.

We use the reduced-form credit risk model proposed by Frey and Schmidt (2012) as vehicle for our analysis. In that model the default times of the reference entity, the protection seller and the protection buyer are conditionally independent given some finite state Markov chain X that models the economic environment. We consider two versions of the model which differ with respect to the amount of information that is available for investors. In the full-information model it is assumed that X is observable so that there are no contagion effects. In the incomplete-information version of the model on the other hand investors observe X in additive Gaussian noise as well as the default history. In that case there is default contagion that is caused by the updating of the conditional distribution of X at the time of default events. An advantage of the setup of Frey and Schmidt (2012) for our purposes is the fact that the the joint distribution of the default times is the same in the two versions of the model. Hence differences in the size of value adjustments or in the performance of collateralization strategies can be attributed purely to the different dynamics of credit spreads (contagion or no contagion) in the two model variants.

In order to compute value adjustments and to measure the performance of collateralization strategies we use the bilateral collateralized credit value adjustment (BCCVA) proposed by Brigo et al. (2014). This credit value adjustment accounts for the form of collateralization strategies and for the credit quality of the contracting parties. Our analysis reveals that the impact of contagion on the size of the BCCVA depends strongly on the relative credit quality of the three parties involved and is hard to predict up front. Results on the performance of different collateralization strategies are more clear-cut: we show that while standard market-value based collateralization strategies provide a good protection against losses due to counterparty risk in the full-information setup, they have problems to deal with the contagious jump in credit spreads at a default of the protection seller. Motivated by these findings, we go on and develop improved collateralization strategies that perform well in the presence of contagion. For our analysis we need to compute the

BCCVA in both model variants. Using Markov chain theory we derive explicit formulas for the BCCVA under full information; in the incomplete-information setup we rely on simulation arguments.

There is by now a large literature on counterparty risk for CDSs. Existing contributions focus mostly on the computation of value adjustments (with and without collateralization) in various credit risk models. Counterparty credit risk and valuation adjustments for uncollateralized CDS are studied by Hull and White (2001), Brigo and Chourdakis (2009), Blanchet-Scalliet and Patras (2008), Lipton and Sepp (2009) and Bao et al. (2012), among others. Counterparty credit risk for collateralized CDS is discussed among others in Bielecki et al. (2011), and Brigo et al. (2014). However, none of these contributions covers the issues discussed in this paper in full. Bielecki et al. (2011) analyze the impact of collateralization on counterparty risk in CDS contracts using the Markov copula model which does not exhibit contagion effects. Brigo et al. (2014) is closest to our contribution: these authors study the impact of contagion on credit value adjustments and on the effectiveness of market-value based collateralization strategies in a Gaussian copula model with stochastic credit spreads. In that model default or event correlation and contagion effects are both driven by the choice of the correlation parameter of the copula. Consequently, it is not possible to disentangle the impact of event correlation and of default contagion on credit value adjustments and on the performance of collateralization strategies. This might be an advantage of our setup. Moreover, Brigo et al. (2014) do not address the issue of designing collateralization strategies that take default contagion into account.

The remainder of the paper is organized in the following way. In Section 2 we discuss the BCCVA of Brigo et al. (2014). In Section 3 we introduce the credit risk model of Frey and Schmidt (2012) that provides the framework for the analysis of the present paper. Section 4 is devoted to the computation of the BCCVA in both model variants. In Section 5 we discuss different collateralization strategies, and in Section 6 we present results from numerical experiments.

2. Bilateral Collateralized Credit Value Adjustment (BCCVA)

In this section we discuss the bilateral collateralized credit value adjustment (BCCVA) proposed in Brigo et al. (2014).

We begin with some notation. Throughout the entire paper we work on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ equipped with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ that fulfills the usual hypotheses. $\mathbb{Q}$ denotes the risk-neutral measure used for pricing, and all expectations are taken with respect to $\mathbb{Q}$. $\mathbb{F}$ is a generic filtration that models the information available to the market participants; we will specify $\mathbb{F}$ when we introduce the credit risk model for our analysis in Section 3. We assume throughout that the short rate $r(u)$ is deterministic and we denote the discount factor from time t to time s by $D(t, s) = e^{-\int_t^s r(u)du}$. The following parties are involved in the CDS contract: the protection buyer, labeled B; the reference identity, labeled R;

the protection seller, labeled S. The default times of these entities are denoted by τ_B, τ_R and τ_S. We introduce the survival indicators $H_t^B := 1_{\{\tau_B > t\}}$, $H_t^R := 1_{\{\tau_R > t\}}$ and $H_t^S := 1_{\{\tau_S > t\}}$ and we put $H := (H^B, H^R, H^S)$. Defaults are observable by assumption so that H is $\mathbb{F}$ adapted and τ_B, τ_R and τ_S are $\mathbb{F}$ stopping times. The first default time is denoted by τ, that is $\tau := \tau_B \wedge \tau_R \wedge \tau_S$. The random variable ξ with values in the set $\{B, R, S\}$ represents the identity of the firm defaulting at τ. Furthermore Rec_B, Rec_R, Rec_S denote the recovery rate and LGD_B, LGD_R, LGD_S the loss given default of of B, R and S, respectively. We assume that recovery rates are constants.

All valuations and cash flows are defined from the perspective of the protection buyer. Therefore positive numbers indicate that a cash flow is received by the protection buyer and negative numbers indicate that a cash flow is received by the protection seller.

Payments of a risk-free CDS. In our context a CDS without counterparty risk, which we call (counterparty-) risk-free CDS, is a CDS where neither the protection seller nor the protection buyer are subject to default risk. For simplicity, we assume that the premium payments are paid continuously. Therefore the sum of all payments in a risk-free CDS from time t to time s discounted to t, is given by:

$$\Pi(t,s) := 1_{\{t < \tau_R \leq s\}} \, \mathrm{LGD}_R \, D(t, \tau_R) - \int_t^s S_R D(t,u) 1_{\{\tau_R > u\}} du, \tag{2.1}$$

where S_R represents the spread of the CDS. In addition we define the time t price of a risk-free CDS with maturity date $T > t$ as the risk-neutral expectation of $\Pi(t,T)$, that is

$$P_t := \mathbb{E}\left(\Pi(t,T) | \mathcal{F}_t\right).$$

Risky CDS and collateralization. In a CDS with counterparty risk, called *risky CDS* below, the protection buyer or the protection seller might default before the maturity of the CDS. Collateralization is a way to limit the potential loss for the surviving party. To keep things simple we assume that the collateral is posted in form of cash and that the collateral earns the risk-free rate of interest $r(s)$. Many collateralization arrangements are in fact of this form, and the additional valuation adjustments that need to be made if the interest rate paid on the collateral differs from the risk-free rate (see for instance Hull and White (2013)) are not central to the issues studied in this paper. Details of the collateralization procedure are stipulated in the credit support annex (CSA) of the contract. Roughly speaking the procedure works as follows. At $t_0 = 0$ a collateral account is opened. Let C_t denote the cash balance in the account at time t. Here $C_t > 0$ means that S has posted the collateral and that B is the collateral taker, whereas $C_t < 0$ means that B has posted the collateral and that S is the collateral taker. The collateral position is updated at discrete time points $t_1, \ldots, t_N \leq T$, for instance daily. At t_1 the collateral taker pays interest on the collateral and the cash balance C_{t_1} is adjusted in reaction to changes in the price of the underlying CDS over $(t_0, t_1]$. This procedure continues up to the maturity of the CDS or until the first default occurs. If $\tau > T$ or if $\tau < T$ and $\xi_1 = R$, the collateral account is closed at the "natural end" of the contract so that $C_t \equiv 0$ for $t \geq \tau \wedge T$. If there is an early default of B or S, that is $\tau \leq T$ and $\xi_1 \in \{B, S\}$, the collateral is used to reduce the loss of the collateral taker and any remaining collateral is returned; details are specified below.

An issue arising in this context is *re-hypothecation*. The collateral taker has unrestricted access to the posted collateral and he may in particular pledge the funds as collateral in other OTC derivative transaction. Hence a part of the collateral is lost at a default of the collateral taker. We denote by Rec'_B and Rec'_S the recovery rate for the return of collateral and by LGD'_B and LGD'_S the corresponding loss given default (assumed constant). Usually the return of collateral is favored to the settlement of other claims in bankruptcy procedures, so that $\mathrm{Rec}_B \leq \mathrm{Rec}'_B$ and $\mathrm{Rec}_S \leq \mathrm{Rec}'_S$. Contracts without re-hypothecation are characterized by $\mathrm{Rec}'_B = \mathrm{Rec}'_S = 1$.

We describe the cash balance in the collateral account by some $\mathbb{F}$-adapted semimartingale $C = (C_t)_{0 \leq t \leq T}$ with RCLL paths, the so-called *collateralization strategy*. For simplicity we assume that interest on the collateral is paid continuously. Since we have assumed that the collateral earns the risk-free rate $r(s)$, from the perspective of the protection buyer collateralization leads to a cumulative cash flow stream given by $C_t - \int_0^t r(s) C_s ds$, $t \leq T$. The discounted value of that cash-flow stream at $t = 0$ equals

$$C_0 + \int_0^T D(0,s) dC_s - \int_0^T D(0,s) r(s) C_s ds = D(0,T) C_T \,, \tag{2.2}$$

where the second equality follows by applying partial integration to $D(0,t)C_t$. Now $C_T = 0$ on $\{\tau > T\}$ and on $\{\tau \leq T\} \cap \{\xi = R\}$. Hence scenarios where neither S nor B default before the end of the underlying CDS can be ignored in the computation of value adjustments for counterparty risk, and it suffices to consider the collateral payments for the case where there is an early default of R or S, that is for $\tau \leq T$ and $\xi \in \{B, S\}$.

Payments at an early default. In order to complete the description of the cash flow stream of a risky CDS we need to specify the payments at an early default of B or S. In that case the surviving party is allowed to charge a *close-out amount* from the defaulting one. According to the ISDA Master Agreement the close-out amount is defined as reasonable estimate of the funds needed to close the position. In this paper we assume that the close-out amount is given by P_τ, the value of the risk-free CDS at the first default time. Note that this choice means that the credit quality of the surviving party is completely neglected in the computation of the close-out amount, which is in line with current market practice. However, there are alternative suggestions in the literature; see for instance Brigo et al. (2012a).

We continue with the description of the payments at an early default. To shorten the exposition we concentrate on the payments in the case where the protection seller defaults first. Note that no additional collateral is posted after the first default. Hence we assume that the amount of collateral available during the bankruptcy process is given by $C_{\tau-}$, the amount of collateral that has been posted immediately prior to τ. This distinction matters if the close-out amount P_t jumps at $t = \tau$, for instance due to contagion effects.

In describing the payments at τ we have to consider four scenarios that differ with respect to the sign of P_τ and of $C_{\tau-}$.

(1) Suppose that $P_\tau > 0$ and that the protection buyer is the collateral taker, that is $C_{\tau-} > 0$. The collateral is used to reduce the loss of the protection buyer. If $C_{\tau-}$ is smaller than P_τ, the protection buyer claims the difference $P_\tau - C_{\tau-}$ from S. However, B will receive only a recovery payment of size $\mathrm{Rec}_S(P_\tau - C_{\tau-})$ in that case. If $C_{\tau-}$ exceeds P_τ, the excess collateral is returned to the protection seller. With the notation $X^+ := \max(X, 0)$ and $X^- := -\min(X, 0)$,[a] in this scenario the overall payment at τ is given by $\mathrm{Rec}_S(P_\tau - C_{\tau-})^+ - (P_\tau - C_{\tau-})^-$.

(2) Suppose next that $P_\tau > 0$ and $C_{\tau-} < 0$, so that the protection seller is the collateral taker. In this situation B is entitled to the repayment of the collateral and to the close-out amount P_τ. However, only a fraction of P_τ and, due to rehypothecation, of $C_{\tau-}$ will be paid to B. Hence in this scenario the overall payment at τ is given by $\mathrm{Rec}_S\, P_\tau - \mathrm{Rec}'_S\, C_{\tau-}$.

(3) Suppose now that $P_\tau < 0$ and that the protection buyer is the collateral taker, that is $C_{\tau-} > 0$. In that case B pays S the close-out amount P_τ and he returns the collateral. Hence from the viewpoint of B, in this scenario the overall payment at τ equals $P_\tau - C_{\tau-}$.

(4) Suppose that $P_\tau < 0$ and that B posted some collateral so that $C_{\tau-} < 0$. If $-C_{\tau-} \leq -P_\tau$ S keeps the collateral and he moreover receives the difference $-(P_\tau - C_\tau)$. Otherwise the excess collateral has to be returned to B, and there might be losses due to re-hypothecation. Hence in this scenario the overall payment at τ equals $\mathrm{Rec}'_S(P_\tau - C_{\tau-})^+ - (P_\tau - C_{\tau-})^-\big)$.

The payments that arise if the protection buyer defaults first, that is if $\xi = B$, can be described in an analogous manner.

The BCCVA. Given a collateralization strategy C, the *bilateral collateralized credit value adjustment* (BCCVA) is defined as difference of the discounted cash-flow stream of the risk-free and the risky CDS. Following Brigo et al. (2014), we denote the latter cash-flow stream by $\Pi^D(t, T, C)$, where D stands for “defaultable”. We thus have

$$\mathrm{BCCVA}(t, T, C) := \mathbb{E}\left(\Pi(t, T) | \mathcal{F}_t\right) - \mathbb{E}\left(\Pi^D(t, T, C) \middle| \mathcal{F}_t\right). \tag{2.3}$$

[a]Note that the convention $X^- := \min(X, 0)$ is used in Brigo et al. (2014).

Using the above description of the payments at an early default it is straightforward to give an explicit formula for $\Pi^D(t,T,C)$. However, in this paper we use an expression for the BCCVA that does not involve Π^D explicitly (see Proposition 2.1 below) so that we omit the formula and refer to Brigo et al. (2014) instead.

By definition the BCCVA measures the difference in value of the cash-flows of a risk-free CDS and a risky CDS. Note that the BCCVA takes the default risk of of S *and* of B into account. The BCCVA thus leads to symmetrical prices in the sense that the adjustment computed from the point of view of the protection buyer equals (with the opposite sign) the adjustment computed from the point of view of the protection seller.

In the sequel we work with the following representation of the BCCVA that is established in Brigo et al. (2014).

Proposition 2.1. *The* BCCVA *can be decomposed as follows*

$$\mathrm{BCCVA}(t,T,C) = \mathrm{CCVA}(t,T,C) - \mathrm{CDVA}(t,T,C), \tag{2.4}$$

where the collateralized credit value adjustment (CCVA) *and the* collateralized debt value adjustment (CDVA) *are given by:*

$$\begin{aligned}\mathrm{CCVA}(t,T,C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=S\}}D(t,\tau)\ (\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ \\ &\qquad\qquad + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+)|\mathcal{F}_t\big)\\ \mathrm{CDVA}(t,T,C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=B\}}D(t,\tau)\ (\mathrm{LGD}_B(C_{\tau-}^- - P_\tau^-)^- \\ &\qquad\qquad + \mathrm{LGD}'_B(P_\tau^+ - C_{\tau-}^+)^-)|\mathcal{F}_t\big).\end{aligned}$$

Comments. 1. The CCVA reflects the possible loss for B due to an early default of S, whereas the CDVA reflects the loss of S due to an early default of B. Consider for instance the case where $\xi = S$. If $P_\tau > 0$, there are two reasons why B might incur a loss: first, the collateral posted by S might be insufficient to cover the close-out amount of the CDS, which leads to a loss of size $\mathrm{LGD}_S(P_\tau - C_{\tau-}^+)^+$; if $C_{\tau-} < 0$ there is moreover a loss due to re-hypothecation given by $\mathrm{LGD}'_S\, C_{\tau-}^-$. If $P_\tau < 0$, B incurs a loss of size $\mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)$ (the loss of the excess collateral caused by re-hypothecation). The overall discounted loss incurred by B is thus given by

$$\begin{aligned}&1_{\{\tau<T\}}1_{\{\xi=S\}}D(t,\tau)\big\{1_{\{P_\tau>0\}}\big(\mathrm{LGD}_S(P_\tau - C_{\tau-}^+)^+ + \mathrm{LGD}'_S\, C_{\tau-}^-\big)\\ &\qquad\qquad + 1_{\{P_\tau<0\}}\,\mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)\big\}\\ &= 1_{\{\tau<T\}}1_{\{\xi=S\}}D(t,\tau)\left\{\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+\right\},\end{aligned}$$

which corresponds to the argument of the CCVA-formula above. In a similar way the CDVA can be interpreted as loss of S on $\{\xi = B\}$.

2. Without collateralization, that is if $C_t \equiv 0$, the value adjustments take the form of options on the risk-free CDS price P with strike price $K = 0$ and random maturity date τ. In that case the terms in (2.4) are labelled BCVA (bilateral credit value adjustment), CVA and DVA.

3. Markets often use a simplified value adjustment formula which implicitly assumes that the survival indicators H^B, H^S and the counterparty-risk free CDS price P are independent stochastic processes, an assumption that is known in the counterparty risk literature as *no wrong-way risk*. For $C_t \equiv 0$ the simplified bilateral credit value adjustment at $t = 0$ is given by

$$\begin{aligned}\mathrm{BCVA}^{\mathrm{indep}} = \mathrm{LGD}_S &\int_0^T \bar{F}_B(s) D(0,s) \mathbb{E}(P_s^+) f_S(s)\, ds \\ - \mathrm{LGD}_B &\int_0^T \bar{F}_S(s) D(0,s) \mathbb{E}(P_s^-) f_B(s)\, ds\,.\end{aligned} \tag{2.5}$$

Here $\bar{F}_S(s) = \mathbb{Q}(\tau_S > s)$ and $f_S(s) = -\bar{F}'(s)$ represent the survival function and the density of τ_S and $\bar{F}_B$ and f_B represent the survival function and the density of τ_B. A derivation of (2.5) is given in Gregory (2010). The independence assumptions underlying the derivation of (2.5) are clearly unrealistic - just think of the case where B, S and R are financial institutions. In Section 6 we therefore study the relation between the "correct" value adjustment (2.4) and the simplified adjustment (2.5). It will turn out that formula (2.5) underestimates the correct value adjustment by a sizeable amount.

3. The Model

Next we give the mathematical description of the model framework that is used in the remainder of this paper. We consider a reduced-form model where τ_R, τ_B and τ_S are conditionally independent, doubly-stochastic random times whose default intensity is driven by a finite-state Markov chain $X = (X_t)_{t \geq 0}$ with state space $S^X = \{1, 2, \ldots, K\}$, generator matrix $W = (w_{ij})_{1 \leq i,j \leq K}$ and initial distribution described by the probability vector π_0 with $\pi_0^k = \mathbb{Q}(X_0 = k)$. Denote by $\mathcal{F}_t^X := \sigma(X_s : s \leq t)$ the filtration generated by X. We assume that for all time points $t_B, t_R, t_S > 0$ one has

$$\mathbb{Q}(\tau_R > t_R, \tau_B > t_B, \tau_S > t_S \mid \mathcal{F}_\infty^X) = \prod_{i \in \{B,R,S\}} \exp\Big(- \int_0^{t_i} \lambda_i(X_s) ds \Big)\,, \tag{3.1}$$

where $\lambda_i : S^X \to \mathbb{R}^+$, $i \in \{B, R, S\}$, are deterministic functions. This definition implies that the default times are independent given the realization of the background process X. In our simulation study we consider the case where $\lambda_B(\cdot)$, $\lambda_R(\cdot)$ and $\lambda_S(\cdot)$ are increasing in x. In that case X can be viewed as an abstract representation of the state of the economy, 1 being the best state (low default probability of all firms) and K the worst state (high default probability of all firms).

For technical reasons we moreover assume that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ supports a d-dimensional standard Brownian motion W which is independent of X and of the survival indicator process H; W is used to model investor information under imperfect observation of X (see below). In the sequel we will consider two variants of the model that differ with respect to the assumptions made on investor information.

The full-information case. Here it is assumed that X is *observable* for investors and we take $\mathbb{F} = \mathbb{F}^O$ with

$$\mathbb{F}^O = \mathbb{F}^H \vee \mathbb{F}^X \vee \mathbb{F}^W , \tag{3.2}$$

where $\mathbb{F}^H$ is the filtration generated by the survival indicators. (The inclusion of $\mathbb{F}^W$ is purely technical and has no impact on the prices of credit derivatives under full information.) It is well-known that for time points $t_R, t_S, t_B > t$ the conditional survival function of (τ_R, τ_S, τ_B) given $\mathcal{F}_t^O$ satisfies

$$\mathbb{Q}(\tau_R > t_R, \tau_S > t_S, \tau_B > t_B \mid \mathcal{F}_t^O) = \prod_{i \in \{R,S,B\}} H_t^i \, \mathbb{E}\Big(\exp\Big(- \int_t^{t_i} \lambda_i(X_s) ds \Big) \mid X_t \Big) . \tag{3.3}$$

Moreover, the process $\lambda_i(X_t)$, $i \in \{R.S, B\}$, is the $\mathbb{F}^O$ default intensity of τ_i, and the pair process (X, H) is Markov. A derivation of these results can be found in Chapter 9 of McNeil, Frey and Embrechts (2005), among others. Formula (3.3) implies in particular that prior to the default of R the price of the risk-free CDS is a function of t and X_t,

$$P_t^O = \mathbb{E}\left(\Pi(t,T) \middle| \mathcal{F}_t^O\right) = H_t^R p^O(t, X_t). \tag{3.4}$$

An explicit formula for the function $p^O(t,k)$ is given in Corollary 4.1 below.

The incomplete-information case. This variant of the model has been studied in detail in Frey and Schmidt (2012). In that paper it is assumed that X is *unobservable* and that investors are confined to a noisy signal of X of the form

$$Z_t := \int_0^t a(X_s) ds + W_t ,$$

where $a : S^X \to \mathbb{R}^d$ is a deterministic function. Hence in this model variant we put $\mathbb{F} = \mathbb{F}^U$ ("unobservable") with

$$\mathbb{F}^U = \mathbb{F}^H \vee \mathbb{F}^Z . \tag{3.5}$$

Note that $\mathbb{F}^U \subseteq \mathbb{F}^O$ by definition.

Under incomplete information the risk-free CDS-price is given by $P_t^U := \mathbb{E}\left(\Pi(t,T) \middle| \mathcal{F}_t^U\right)$. P_t^U can be computed by projecting the full-information price $H_t^B p^O(t, X_t)$ (see (3.4)) on $\mathbb{F}^O$. We get, as $\mathbb{F}^U \subseteq \mathbb{F}^O$,

$$P_t^U = \mathbb{E}\left(\Pi(t,T) \middle| \mathcal{F}_t^U\right) = \mathbb{E}\left(\mathbb{E}\left(\Pi(t,T) \middle| \mathcal{F}_t^O\right) \middle| \mathcal{F}_t^U\right) = H_t^R \mathbb{E}\left(p^O(t, X_t) \middle| F_t^U\right) . \tag{3.6}$$

Define the conditional probabilities

$$\pi_t^k := \mathbb{Q}(X_t = k \mid \mathcal{F}_t^U), \ 1 \le k \le K, \ \text{ and let } \pi_t := (\pi_t^1, \ldots, \pi_t^K)^\top . \tag{3.7}$$

With this notation (3.6) can be written more succinctly as

$$P_t^U = H_t^R \sum_{k \in S^X} \pi_t^k p^O(t, k) . \tag{3.8}$$

Comments. Note that relation (3.8) involves conditional probabilities with respect to the pricing measure $\mathbb{Q}$.

In Proposition 4.2 below we will show that the $\mathbb{Q}$-dynamics of π_t can be described by a K-dimensional SDE system. From this system we may in particular derive an explicit representation for the contagion effects under incomplete information.

In the practical application of the model the process Z is considered as abstract source of information and the current value of π is calibrated from observed prices of traded credit derivatives; see Section 6.1 below.

In both model variants the unconditional joint survival function of τ_R, τ_B and τ_S is given by

$$\mathbb{Q}(\tau_R > t_R, \tau_B > t_B, \tau_S > t_S) = \mathbb{E}\Big(\prod_{i \in \{B,R,S\}} \exp\Big(- \int_0^{t_i} \lambda_i(X_s) ds \Big) \Big),$$

so that the distributions of (τ_B, τ_R, τ_S) coincides in both versions of the model. Therefore any differences in the BCCVA or in the performance of collateralization strategies can be attributed to the different dynamics of CDS spreads. For illustrative purposes we plot typical trajectories of CDS spreads in both model variants in Figure 1.

4. Computation of the BCCVA

4.1. *The case of the full-information model*

In order to evaluate the BCCVA-formula (2.4) we need to determine the joint distribution of τ, ξ and X_τ. This is done in Proposition 4.1 below. The proof of this result relies on the observation that the distribution of the triple (τ, ξ, X_τ) can be expressed as first entry time of the processes (X, H^R) and (X, H) into specific sets. Since in our setting these processes form a finite-state Markov chain one can use Markov-chain theory to derive their distribution. In order to give precise results, we need to specify the generator matrix of these Markov chains.

For this we assume that the states are ordered in the *inverse lexicographic order*. According to this order a vector $(x_1, \ldots, x_n)$ is smaller than $(y_1, \ldots, y_n)$ if $x_n < y_n$ or if there is some $k < n$ with $x_{l+1} = y_{l+1}$ for $l \in \{k, \ldots, n-1\}$ and with $x_k < y_k$. For example, in the case $K = 2$ the states of the process (X, H^R) are ordered in the following way:

$$(1,0) < (2,0) < (1,1) < (2,1).$$

The transition rate $q_{y,z}$ of (X, H^R) from a state $y = (y_1, y_2)$ to the state $z = (z_1, z_2)$ is given by:

$$q_{y,z} = \begin{cases} w_{y_1 z_1} & \text{if } y_1 \neq z_1 \text{ and } y_2 = z_2 \\ \lambda_R(y_1) & \text{if } y_1 = z_1, y_2 = 0 \text{ and } z_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence the generator of the process (X, H^R) can be represented by the matrix

$$Q := \begin{pmatrix} W - \Lambda_R & \Lambda_R \\ 0 & W \end{pmatrix}.$$

Here $\Lambda_R = \mathrm{diag}(\lambda_R(1), \dots, \lambda_R(K))$ denotes a diagonal matrix with entries on the main diagonal given by the elements of the vector λ_R. The transition rates and the generator of (X, H) can be determined by analogous considerations.

Proposition 4.1. *Let $t < s$ and $k \in S^X$. Then the following statements hold:*

(a) The distribution of τ_i with $i \in \{B, R, S\}$ satisfies

$$\mathbb{Q}\left(\tau_i \le s \middle| X_t = k, H_t^i = 0\right) = 1 - e_k^\top e^{Q_i(s-t)} \boldsymbol{1}^K .$$

Here $Q_i := W - \Lambda_i$ where $\Lambda_i = \mathrm{diag}(\lambda_i(1), \dots, \lambda_i(K))$, $\boldsymbol{1}^K = (1, \dots, 1)^\top$ is a column vector of dimension K and e_k denotes the kth unit vector in $\mathbb{R}^K$.

(b) The distribution of the first-to-default time τ can be computed as:

$$\mathbb{Q}\left(\tau \le s | X_t = k, H_t = 0\right) = 1 - e_k^\top e^{Q_{(1)}(s-t)} \boldsymbol{1}^K,$$

where we defined $Q_{(1)} := W - \sum_{j \in \{B,R,S\}} \Lambda_j$.

(c) The probability that obliger $i \in \{B, R, S\}$ defaults first and before time s is:

$$\mathbb{Q}\left(\tau_i \le s, \xi = i | X_t = k, H_t = 0\right) = e_k^\top Q_{(1)}^{-1} \left(e^{Q_{(1)}(s-t)} - I\right) \Lambda_i \boldsymbol{1}^K.$$

Here $Q_{(1)}^{-1}$ is the inverse of $Q_{(1)}$.

(d) The probability that obliger $i \in \{B, R, S\}$ defaults first and that at default the Markov chain is in the state l equals

$$\mathbb{Q}\left(X_\tau = l, \tau_i \le s, \xi = i | X_t = k, H_t = 0\right) = e_k^\top Q_{(1)}^{-1} \left(e^{Q_{(1)}(s-t)} - I\right) \Lambda_i e_l.$$

The proof of this result can be found in Appendix A. The first two claims are well-known and have been derived among others by Graziano and Rogers (2009), see also Herbertsson (2011). However we include their proof for the convenience of the reader. Statements c) and d) on the other hand have to the best of our knowledge not appeared previously in the literature.

Using Proposition 4.1 the following well-known formula for the price of a risk-free CDS can be deduced.

Corollary 4.1 (Risk-free CDS price under full-information). *The price P_t^O of a risk-free CDS with generic swap spread S on R given that $X_t = k$ and $\tau_R > t$ is equal to $1_{\{\tau_R > t\}} p^O(t, k)$, where the function $p^O : [0, T] \times S^X \to \mathbb{R}$ is given by*

$$p^O(t, k) = \left(-\mathrm{LGD}_R\, e_k^\top Q_R - S e_k^\top\right) (Q_R - rI)^{-1} \left(e^{(Q_R - rI)(T-t)} - I\right) \boldsymbol{1}^K.$$

Here $Q_R = W - \mathrm{diag}(\lambda_R(1), \dots, \lambda_R(K))$, see Proposition 4.1 (a). Moreover, the price of a CDS at $t = 0$ is

$$P_0^O = \left(-\mathrm{LGD}_R\, \pi_0^\top Q_R - S \pi_0^\top\right) (Q_R - rI)^{-1} \left(e^{(Q_R - rI)(T-t)} - I\right) \boldsymbol{1}^K.$$

Below we will see that for a suitable function $g\colon [0,T]\times S^X \to \mathbb{R}$, collateralization strategies of the form $C_t = g(t, X_t)$ are optimal in the full-information model. For a generic strategy of this form, Theorem 4.1(d) gives the following semi-closed formula for the BCCVA.

Corollary 4.2 (BCCVA formula under full information). *Assume that for $t < \tau$ the collateralization strategy is of the form $C_t = g(t, X_t)$. Then, given $X_t = j$, the* CCVA *and* CDVA *are given by:*

$$\begin{aligned}
\mathrm{CCVA}_t = \sum_{k\in\{1,\dots,K\}} \int_t^T D(t,s) \Big(& \mathrm{LGD}_S \left(p^O(s,k)^+ - g(s,k)^+\right)^+ \\
& + \mathrm{LGD}'_S \left(g(s,k)^- - p^O(s,k)^-\right)^+ \Big) f^S_{j,k}(s)\, ds \\
\mathrm{CDVA}_t = \sum_{k\in\{1,\dots,K\}} \int_t^T D(t,s) \Big(& \mathrm{LGD}_B \left(g(s,k)^- - p(s,k)^-\right)^- \ , \\
& + \mathrm{LGD}'_B \left(p(s,k)^+ - g(s,k)^+\right)^- \Big) f^B_{j,k}(s)\, ds\,.
\end{aligned}$$

Here the functions $f^i_{j,k}$, $i \in \{B, S\}$, are given by

$$f^i_{j,k}(s) := \frac{d}{ds}\mathbb{Q}(\tau \le s, \xi = i, X_\tau = k \mid X_t = j, H_t = 0) = 1_{\{\tau_i > t\}} e_j^\top e^{Q_{(1)}(s-t)} \Lambda_i e_k.$$

4.2. *The* BCCVA *in the incomplete-information model*

In this section we discuss the computation of the BCCVA under incomplete information. We begin with a formula for the risk-free CDS price. By combining (3.8) and Corollary 4.1 we obtain

Corollary 4.3 (risk-free CDS prices under incomplete information). *Given that $\{\tau_R > t\}$ the value P^U_t of a risk-free CDS at time t equals*

$$P^U_t = p^U(t, \pi_t) := \left(-\mathrm{LGD}_R\, \pi_t^\top Q_R - S\pi_t^\top\right) (Q_R - rI)^{-1} \left(e^{(Q_R - rI)(T-t)} - I\right) \boldsymbol{1}^K.$$

Note that for $t = 0$ one has $P^O_0 = P^U_0$; this equality reflects of course the fact that the unconditional distributions of the default times coincide in the two model variants.

Under incomplete information the BCCVA is essentially the value of a portfolio of options on the price P^U of the risk-free CDS. Since P^U_t is a function of π_t, in order to compute the BCCVA one thus needs the form of the dynamics of the process π, and we now recall the relevant results from Frey and Schmidt (2012). We begin with some notation. We denote the $\mathbb{Q}$-optional projection of a process $G = (G_t)_{t\in[0,T]}$

with respect to $\mathbb{F}^U$ by $\widehat{G}$, that is $\widehat{G}_t = \mathbb{E}\left(G_t \middle| \mathcal{F}_t^U\right)$. In particular,

$$(\widehat{\lambda_i})_t = \mathbb{E}\left(\lambda_i(X_t)\middle|\mathcal{F}_t^U\right) = \sum_{k=1}^K \lambda_i(k)\pi_t^k, \quad i \in \{B, R, S\},$$
$$\widehat{a}_t = \mathbb{E}\left(a(X_t)\middle|\mathcal{F}_t^U\right) = \sum_{k=1}^K a(k)\pi_t^k\,.$$

Using the Levy-characterization of Brownian motion it is easily seen that

$$\mu_t = (\mu_t^1, \dots, \mu_t^d) \text{ with } \mu_t^i = Z_t^i - \int_0^t (\widehat{a}_i)_s\, ds$$

is a $\mathbb{F}^U$-Brownian motion. In the literature on stochastic filtering such as Bain and Crisan (2009), $(\mu_t)_{0\le t\le T}$ is known as *innovations process.* Moreover, it is well-known that $\widehat{\lambda}_i$ is the $\mathbb{F}^U$ default intensity of firm i, that is for $i \in \{B, R, S\}$,

$$M_t^i := 1_{\{\tau_i \le t\}} - \int_0^t H_s^i\ (\widehat{\lambda}_i)_s\, ds$$

is an $\mathbb{F}^U$ martingale, see for instance Chapter 2 of Bremaud (1981).

Proposition 4.2 (Kushner-Stratonovich-equation). *The process π is the unique solution of the K-dimensional SDE system*

$$d\pi_t^k = \sum_{i=1}^K w_{ik}\pi_t^i dt + \sum_{j\in\{R,B,S\}} \left(\gamma_j^k(\pi_{t-})\right)^\top dM_t^j + \left(\alpha^k(\pi_t)\right)^\top d\mu_t\,,\ k = 1, \dots, K\,,$$

where

$$\gamma_j^k(\pi_t) = \pi_t^k\left(\frac{\lambda_j(k)}{\sum_{i=1}^K \lambda_j(i)\pi_t^i} - 1\right) \text{ for } 1 \le j \le K \text{ and}$$
$$\alpha^k(\pi_t) = \pi_t^k\left(a(k) - \sum_{j=1}^K \pi_t^j a(j)\right).$$

The proposition shows that the process π exhibits jump-diffusion dynamics. In particular, π jumps at default times and the jump height of π_t^k at the default of firm j is equal to $\gamma_j^k(\pi_{\tau_j-})$. Furthermore using the proposition we can compute the size of the information-induced contagion effects: the jump in the $\mathbb{F}^U$-default intensity of firm i at the default of firm j, $j \ne i$ equals

$$(\widehat{\lambda}_i)_{\tau_j} - (\widehat{\lambda}_i)_{\tau_j-} = \sum_{k=1}^K \lambda_i(k)\pi_{\tau_j-}^k\left(\frac{\lambda_j(k)}{\sum_{l=1}^K \lambda_j(l)\pi_{\tau_j-}^l} - 1\right) = \frac{\mathrm{cov}^{\pi_{\tau_j-}}(\lambda_i, \lambda_j)}{E^{\pi_{\tau_j-}}(\lambda_j)}. \tag{4.1}$$

An inspection of the formula (4.1) shows the following.

- Contagion effects are inversely proportional to the instantaneous default risk of the defaulting entity (firm j): a default of an entity with a better credit quality comes as a bigger surprise and the market impact is larger.

- Contagion effects are proportional to the covariance of the default intensities $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$ under the 'a-priori distribution' $\pi_{\tau_j -}$. In particular, contagion effects are relatively high if the firms have similar characteristics in the sense that the functions $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$ are (almost) linearly dependent.

Proposition 4.2 indicates a method to simulate a trajectory of π. The following general approach is suggested in Frey and Schmidt (2012).

(1) Generate a trajectory of the Markov chain X.
(2) Generate for the trajectory of X constructed in (i) a trajectory of the default indicator H and the noisy information Z.
(3) Solve the system of SDEs numerically, for instance via a Euler-Maruyama type method.

We close this section with a theoretical result on the relationship between the un-collateralized value adjustments in the two versions of the model.

Proposition 4.3. *Assume that the CDS contract is un-collateralized, i.e.* $C_t \equiv 0$. *Then the following relationships hold:*

$$\mathrm{CVA}_0^O \geq \mathrm{CVA}_0^U \;\; \textit{and} \;\; \mathrm{DVA}_0^O \geq \mathrm{DVA}_0^U \,.$$

Proof. We begin with the CVA. Using the definition of the CVA, Jensen's inequality and the relation $P_\tau^U = \mathbb{E}(P_\tau^O \mid \mathcal{F}_\tau^U)$, we get

$$\begin{aligned}
\mathrm{CVA}_0^O &= \mathrm{LGD}_S \, \mathbb{E}\left(1_{\{t<\tau\leq T\}} 1_{\{\xi=S\}} \left(P_\tau^O\right)^+\right) \\
&= \mathrm{LGD}_S \, \mathbb{E}\left(1_{\{t<\tau\leq T\}} 1_{\{\xi=S\}} \mathbb{E}\left(\left(P_\tau^O\right)^+ \middle| \mathcal{F}_\tau^U\right)\right) \\
&\geq \mathrm{LGD}_S \, \mathbb{E}\left(1_{\{t<\tau\leq T\}} 1_{\{\xi=S\}} \left(\mathbb{E}\left(P_\tau^O \middle| \mathcal{F}_\tau^U\right)\right)^+\right) \\
&= \mathrm{LGD}_S \, \mathbb{E}\left(1_{\{t<\tau\leq T\}} 1_{\{\xi=S\}} (P_\tau^U)^+\right),
\end{aligned}$$

and the last line is obviously equal to CVA_0^U. A similar reasoning applies to the DVA. □

The overall relation of the BCVA in the two model variants is in general unclear, since the BCVA is the difference of the CVA and DVA. If B is of a much higher credit quality than S, the DVA is almost zero and we have the relation $\mathrm{BCVA}^O \geq \mathrm{BCVA}^U$. Similarly, if S is of a much higher credit quality than B, one has $\mathrm{BCVA}^O \leq \mathrm{BCVA}^U$.

The intuition underlying (the proof of) the result is as follows: First, the CVA is the price of an option on the risk-free CDS price with exercise price $K = 0$. Moreover, since $P_\tau^U = \mathbb{E}(P_\tau^O \mid \mathcal{F}_\tau^U)$ the variance of P_τ^U is smaller than the variance of P_τ^O. Since the price of an option increases with increasing variance of the distribution of the underlying asset value, we get that $\mathrm{CVA}^O \geq \mathrm{CVA}^U$.

5. Collateralization strategies

Standard collateralization strategies. We consider among others the following collateralization strategies. *No collateralization* corresponds to the strategy $C_t \equiv 0$. The *threshold-collateralization* strategy with initial margin γ and thresholds M_1, $M_2 \geq 0$, labeled C^{γ,M_1,M_2}, is given by

$$C_t^{\gamma,M_1,M_2} := \gamma + (P_t - M_1)1_{\{P_{t-} > M_1\}} + (P_t + M_2)1_{\{P_t < -M_2\}} \; \forall t \in [0, T \wedge \tau).$$

This strategy is used if B and S want to protect themselves against severe losses, while accepting the possibility of small losses in order to simplify the collateralization process. At the beginning of the contract an initial payment of collateral of size γ takes place, which is a crude device to account for contagion effects. Additional collateral is only posted if the exposure of one entity exceeds some threshold (M_1 in case of B and M_2 in case of S). Threshold collateralization is quite popular in practice, see Gregory (2010). However, the choice of γ in practice is often based on rules of thumb (compare ICMAs European Repo Council (2012) for Repos), possibly reducing the effectiveness of this strategy. For $\gamma = M_1 = M_2 = 0$ we obtain the special case of *market-value collateralization* C^{market} with $C_t^{\text{market}} = P_t$.

Improved collateralization strategies. In the following we study collateralization strategies that attempt to reduce the overall counterparty-risk exposure of the contracting parties. We use the CCVA to measure the exposure to counterparty risk of B and the CDVA to measure the exposure of S. B and S have obviously conflicting interests: B prefers a collateralization strategy where S posts a large amount of collateral and B posts no collateral and vice versa for B. In order to balance these conflicting interests we consider an $\mathbb{F}$ adapted collateralization strategy C to be optimal if it minimizes the following functional

$$\begin{aligned} m(C) &:= \text{CCVA}_0 + \text{CDVA}_0 && (5.1)\\ &= \mathbb{E}\left(1_{\{\tau<T\}}1_{\{\xi=S\}}D(t,\tau)(\text{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ + \text{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+)\right) && (5.2)\\ &+ \mathbb{E}\left(1_{\{\tau<T\}}1_{\{\xi=B\}}D(t,\tau)(\text{LGD}_B(C_{\tau-}^- - P_\tau^-)^- + \text{LGD}'_B(P_\tau^+ - C_{\tau-}^+)^-)\right) && (5.3) \end{aligned}$$

In the full-information case we let $\mathbb{F} = \mathbb{F}^O$ and $P_\tau = P_\tau^O$; in the incomplete-information case we let $\mathbb{F} = \mathbb{F}^U$ and $P_\tau = P_\tau^U$.

The analysis of the *full-information model* is straightforward. In that case the market value $(P_t^O)_{t\geq 0}$ is continuous at τ_B respectively at τ_S. Therefore counterparty risk can be eliminated completely by choosing the market-value strategy $C_t^{\text{market}} = P_t^O = p^O(t, X_t)$, $t < \tau$, that is $m(C^{\text{market}}) = 0$. Note that this result holds in all credit risk models where the risk-free CDS price does not jump at τ_S or τ_R, that is for $\Delta P_{\tau_S} = \Delta P_{\tau_R} = 0$, and thus in particular in all models with conditionally independent defaults.

Optimal strategies under incomplete information. The situation is more involved in the incomplete-information model. In that case the jump of π at τ

leads to a jump in the market value P_t^U of the CDS at $t = \tau$ and the collateral position cannot be adjusted at that point. Hence for the market value strategy $C_t^{\text{market}} = P_t^U = p^U(t, \pi_t)$, $t < \tau$ holds that $m(C^{\text{market}}) > 0$.

We therefore need to work a bit more in order to find an optimal strategy under incomplete information. As a first step we simplify the functional m by conditioning on $\mathcal{F}_{\tau-}$. It is well-known that τ is $\mathcal{F}_{\tau-}$ measurable and that for any predictable process L the random variable L_τ is $\mathcal{F}_{\tau-}$ measurable; see Protter (2005), Sec III.2. Moreover, for $j \in \{R, B, S\}$ it holds that

$$\mathbb{Q}\left(\xi = j | \mathcal{F}_{\tau-}\right) = \frac{(\widehat{\lambda_j})_{\tau-}}{\sum_{i \in \{B,R,S\}} (\widehat{\lambda}_i)_{\tau-}} =: d_j(\pi_{\tau-}). \tag{5.4}$$

We begin with the CCVA component of m. By conditioning on $\mathcal{F}_{\tau-}$ we get that (5.2) equals

$$\mathbb{E}\left(1_{\{\tau \le T\}} D(t,\tau) \mathbb{E}\left(1_{\{\xi = S\}} \left(\mathrm{LGD}_S (P_\tau^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S (C_{\tau-}^- - P_\tau^-)^+\right) \big| \mathcal{F}_{\tau-}\right)\right) \tag{5.5}$$

In the sequel we use the notation

$$x_S := x_S(\tau, \pi_{\tau-}) = p^U\Big(\tau, \pi_{\tau-} + \mathrm{diag}\left(\gamma_S^1(\pi_{\tau-}), \ldots, \gamma_S^K(\pi_{\tau-})\right)\Big) \tag{5.6}$$

to denote the price of the CDS immediately after the default of S; similarly, $x_B := x_B(\tau, \pi_{\tau-})$ denotes the price of the CDS immediately after the default of B. Now note that $1_{\{\xi=S\}} P_\tau^+ = x_S^+$. Hence, using (5.4), the inner conditional expectation in (5.5) is given by $d_S\big(\mathrm{LGD}_S(x_S^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S(C_{\tau-}^- - x_S^-)^+\big)$, and (5.5) equals

$$\mathbb{E}\left(1_{\{\tau \le T\}} D(t,\tau) d_S \left(\mathrm{LGD}_S(x_S^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S(C_{\tau-}^- - x_S^-)^+\right)\right).$$

Similarly we get that (5.3), the CDVA component of m, is equal to

$$\mathbb{E}\left(1_{\{\tau \le T\}} D(t,\tau) d_B \left(\mathrm{LGD}_B(C_{\tau-}^- - x_B^-)^- + \mathrm{LGD}'_B(x_B^+ - C_{\tau-}^+)^-\right)\right).$$

Define now the 'infinitesimal loss function' by

$$\begin{aligned} l(t,\pi,c) = {} & d_S(\pi) \left((\mathrm{LGD}_S(x_S(t,\pi)^+ - c^+)^+) + \mathrm{LGD}'_S(c^- - x_S(t,\pi)^-)^+\right) \\ & + d_B(\pi) \left((\mathrm{LGD}_B(c^- - x_B(t,\pi)^-)^- + \mathrm{LGD}'_B(x_B(t,\pi)^+ - c^+)^-\right). \end{aligned}$$

The above computations show that $m(C)$ can be written in the form $m(C) = \mathbb{E}\left(D(t,\tau) l(\tau, \pi_{\tau-}, C_{\tau-})\right)$. Now suppose that we find an $\mathbb{F}^U$-adapted RCLL-process C^* such that a.s.

$$C_t^*(\omega) \in \mathrm{argmin}\{l(t, \pi_t(\omega), c) \colon c \in \mathbb{R}\}.$$

Then C^* is an optimal collateralization strategy - a minimizer of $m(\cdot)$ - in the incomplete-information model. This leads to the following proposition.

Proposition 5.1. *Denote by $x_S = x_S(t, \pi_t)$ and $x_B = x_B(t, \pi_t)$ the risk-free CDS price at time t given $\tau = t, \xi = S$ respectively $\tau = t, \xi = B$ (see (5.6)) and let*

$$d_S = d_S(\pi_t) = \frac{(\widehat{\lambda_S})_t}{\sum_{i \in \{B,R,S\}} (\widehat{\lambda}_i)_t} \quad \text{and } d_B = d_B(\pi_t) = \frac{(\widehat{\lambda_B})_t}{\sum_{i \in \{B,R,S\}} (\widehat{\lambda}_i)_t}.$$

Then an $\mathbb{F}^U$-adapted RCLL process C^* is an optimal collateralization strategy under incomplete information if and only if the following relations hold $\mathbb{Q}$-a.s. for $t < \tau$:

$$C_t^* = \begin{cases} x_S & \text{if } 0 \le x^B \le x^S, \mathrm{LGD}'_B\, d_B < \mathrm{LGD}_S\, d_S \\ x_B & \text{if } 0 \le x^B \le x^S, \mathrm{LGD}'_B\, d_B > \mathrm{LGD}_S\, d_S \\ x_S & \text{if } x_B \le x_S \le 0, \mathrm{LGD}_B\, d_B < \mathrm{LGD}'_S\, d_S \\ x_B & \text{if } x_B \le x_S \le 0, \mathrm{LGD}_B\, d_B > \mathrm{LGD}'_S\, d_S \\ \mathrm{argmin}\{l(\tau, \pi_t, c) \colon c = x_B, 0, x_S\} & \text{if } x_B < 0 < x_S \end{cases}$$

$$C_t^* \in \begin{cases} [x_B, x_S] & \text{if } 0 \le x_B \le x_S, \mathrm{LGD}'_B\, d_B = \mathrm{LGD}_S\, d_S \\ [x_B, x_S] & \text{if } x_B \le x_S \le 0, \mathrm{LGD}_B\, d_B = \mathrm{LGD}'_S\, d_S \\ [x_S, x_B] & \text{if } x_S \le x_B\,. \end{cases}$$

In particular for any such strategy it holds that $C_t^* \in [\min\{x^S, x^B\}, \max\{x^S, x^B\}]$ $\forall t$ and that $l(t, \pi, C_t^*) = 0$ for $x^S \le x^B$.

Proof. The proof relies on the preceding arguments. In order to find an optimal strategy we have to find the minimizers of the function $c \mapsto l(\tau, \pi, c)$. This is a piecewise linear function, which converges to ∞ for $c \to \pm\infty$ and fixed t, π. Therefore a minimum, exists; it can be found by a case-by-case analysis. Consider for instance the case $0 < x_B < x_S$. In that case l takes the form

$$l(t, \pi, c) = (\mathrm{LGD}_S (x_S - c^+)^+ + \mathrm{LGD}'_S\, c^-) d_S + (\mathrm{LGD}'_B (x_B - c^+)^-) d_B,$$

and l is decreasing in the interval $(-\infty, x_B]$ and increasing in $[x_S, \infty)$. Therefore the optimal c lies in $[x_B, x_S]$. For $c \in [x_B, x_S]$, l is given by:

$$l(\tau, \pi_{\tau-}, c) = c(\mathrm{LGD}'_B\, d_B - \mathrm{LGD}_S\, d_S) + \mathrm{LGD}_S\, x_S d_S - \mathrm{LGD}'_B\, x_B d_B.$$

Therefore the result follows. The other scases are dealt with by a similar analysis. $\square$

We will see below that this optimal collateralization strategy reduces counterparty risk by a large amount compared to standard market-value collateralization. However, if $\mathbb{Q}\big(x^B(t, \pi_t) > x^S(t, \pi_t)\big) > 0$ there remains some risk, that is $m(C^*) > 0$. This remaining risk is due to the fact that in an inhomogeneous portfolio the size of the contagion effects at τ depends on the identity of the defaulting firm which cannot be predicted upfront given the information contained in $\mathcal{F}_{\tau-}$.

Model-independent strategies. The optimal strategy derived in Proposition 5.1 depends on d_B, d_S, and, most importantly, on the market value x_S and x_B of the risk-free CDS after the default of S or B and hence on the size of contagion effects. While these quantities can be computed within a specific reduced-form credit risk model with contagion such as the model of Frey and Schmidt (2012) or the model considered by Brigo et al. (2012a), they do depend on the structure of the model and on the parameter values used. It is therefore of interest to develop a 'model-independent' version of C^*.

For this one needs to estimate d_B, d_S, x_B and x_S. Given the well-known rule of thumb that the CDS spread of a firm is roughly equal to the product of its default intensity and loss given default, in view of the definition of d_B and d_S in (5.4) it is natural to estimate d_B and d_S by

$$\widehat{d_S} = \frac{\frac{S_S}{\mathrm{LGD}_S}}{\frac{S_B}{\mathrm{LGD}_B} + \frac{S_S}{\mathrm{LGD}_S} + \frac{S_R}{\mathrm{LGD}_R}} \quad \text{and} \quad \widehat{d_B} = \frac{\frac{S_S}{\mathrm{LGD}_S}}{\frac{S_B}{\mathrm{LGD}_B} + \frac{S_B}{\mathrm{LGD}_B} + \frac{S_R}{\mathrm{LGD}_R}}\,, \tag{5.7}$$

where S_B, S_R and S_S represent the fair CDS spread for B, R and S observed in the market. Estimating x_S and x_B is less straightforward. Here one could use ad-hoc assumptions, based on the analysis of historical contagious events. Alternatively we propose to use our results on contagion effects in the Frey and Schmidt (2012)-model (see relation (4.1)). This gives

$$x_B(t, \pi_t) \approx P_t + \mathrm{LGD}_R(\Delta\widehat{\lambda_R})\mid_{t=\tau_B} = P_t + \mathrm{LGD}_R\,\frac{\mathrm{cov}^{\pi_t}(\lambda_R, \lambda_B)}{(\widehat{\lambda_R})_t}$$

Now we suggest to proxy $\widehat{\lambda_R})_t$ by $S_{R,t}/\,\mathrm{LGD}_R$ ($S_{R,t}$ the CDS spread observed at time t and $\mathrm{cov}^{\pi_t}(\lambda_R, \lambda_B)$ by $(\mathrm{LGD}_R\,\mathrm{LGD}_B)^{-1}\widehat{\mathrm{cov}}_t(S_R, S_B)$ where $\widehat{\mathrm{cov}}_t(S_R, S_B)$ is an estimate of the time series covariance of the observed CDS spreads at time t obtained for instance by some exponentially weighted historical average. This gives the estimators

$$(\widehat{x_B})_t = P_t + \frac{\mathrm{LGD}_R}{\mathrm{LGD}_B}\frac{\widehat{\mathrm{cov}}_t(S_R, S_B)}{S_{R,t}} \quad \text{and} \quad (\widehat{x_S})_t = P_t + \frac{\mathrm{LGD}_R}{\mathrm{LGD}_S}\frac{\widehat{\mathrm{cov}}_t(S_R, S_S)}{S_{R,t}}\,. \tag{5.8}$$

Note that the proposed estimators for d_B, d_S, x_B and x_S can be computed directly from a time series of observed CDS spreads without making reference to a particular model. In the next section we compare the performance of the model-independent refined strategy with the performance of market-value collateralization on the one hand and with the optimal strategy on the other.

6. Numerical Experiments

In this section we discuss results from a number of numerical experiments.

6.1. *Setup and Calibration*

We considered a Markov chain X with $K = 8$ states. It was assumed that X exhibits next-neighbor dynamics (X_t jumps only to $X_t \pm 1$), so that only the values on the main diagonal and on the first off-diagonal of the generator matrix W differ from zero. During the simulation analysis we set the entries on the off-diagonal equal to 0.25, meaning that the Markov chain jumps on average every second year. We have put the short-rate equal to $r = 0.015$. Throughout the study it was assumed that $\mathrm{Rec}_B = \mathrm{Rec}_S = \mathrm{Rec}_R = 0.5$ and $\mathrm{Rec}'_B = \mathrm{Rec}'_S = 0.75$. [b] We calibrated the model

[b]Following a suggestion of the referee who was rightly concerned with the robustness of our findings we ran our simulations also with different forms of the generator matrix W. This led to

to the risk-free CDS spreads and default correlations for R, B and S given in Table 6.1. We considered five different scenarios, labeled *Base; Base 2; Symmetric; Risky protection buyer; Risky protection seller*. These scenarios differ mainly with respect to the relative riskiness of the firms involved in the CDS contract; their choice serves to illustrate the impact of the relative riskiness of the different firms on credit value adjustments. The fair CDS spreads (in basis points) and default correlations (in percentage points) corresponding to these scenarios can be found in Table 6.1.

Table 1. Risk scenarios: CDS-spreads in base points, default correlations in percentage points.

Name of scenario	B	R	S	ρ_{BR}	ρ_{BS}	ρ_{RS}
Base	50	1000	500	2.0	1.5	5.0
Base2	500	1000	50	5.0	1.5	2.0
Symmetric	500	1000	500	5.0	3.0	5.0
Risky PB	1000	500	50	5.0	2.0	1.5
Risky PS	50	500	1000	1.5	2.0	5.0

In this context model model calibration amounts to finding the initial distribution of the Markov chain π_0 and the parameters $\lambda_B, \lambda_R, \lambda_S$. For calibration purposes we used a modification of the algorithm presented in Frey and Schmidt (2012); since the focus of this paper is not on model calibration we omit the details. All in all the calibration procedure performed well, with very small errors for CDS spreads (absolute errors are less than 0.5 bp) and acceptable results for default correlations (relative errors are around 3%). The calibrated values of π_0 and of λ_B, λ_R and λ_S can be found in the appendix, Table 2. Note in particular that the calibrated functions $\lambda_B(\cdot)$, $\lambda_S(\cdot)$ and $\lambda_R(\cdot)$ are increasing in x. In the incomplete-information model we moreover need to choose the parameters $a(1), \ldots, a(K)$. We took $a = c * b$, where $b = [-1.75, -1.25, -0.75, -0.25, 0.25, 0.75, 1.25, 1.75]$ and where $c \geq 0$ If not mentioned otherwise, c was taken equal to one.

6.2. *Results for the un-collateralized case*

The main findings regarding the qualitative behavior of the CVA and the DVA in the un-collateralized case can be summarized as follows.

a) The size of the credit value adjustments depends largely on the relative riskiness of the firms. In particular, the CVA is comparatively high if the *first-to-default probability* $\mathbb{Q}(\tau \leq T, \xi = S)$ is relatively large; similarly, the DVA is comparatively high if $\mathbb{Q}(\tau \leq T, \xi = B)$ is relatively large. This can be seen by comparing the size of the value adjustments for the Base and Base2 scenarios or the RiskyPB and the RiskyPS scenarios in Table 5: as shown in Table 4, $\mathbb{Q}(\tau \leq T, \xi = S)$ is relatively large in the Base and the RiskyPS scenarios, leading to a high CVA; similarly,

qualitatively similar results. We do not report these results here as we do not want to overload the paper with numbers.

$\mathbb{Q}(\tau \le T, \xi = B)$ is relatively large in the Base2 and the RiskyPB scenarios, leading to a high DVA. Note that the first-to-default probabilities are identical in both versions of the model. They are largely driven by the (relative) riskiness of the three firms as given by the risk-free CDS spread in the three scenarios.

b) We have $\mathrm{CVA}^U < \mathrm{CVA}^O$ and $\mathrm{DVA}^U < \mathrm{CVA}^O$, as predicted by Proposition 4.3. The differences between the model variants decreases with decreasing observation noise, that is for higher values of the parameter c in the definition of the function a, as can be seen by inspection of Table 6.

c) The conditional default probability of the reference entity given an early default of the protection seller is much higher than the unconditional default probability of R (so-called wrong-way risk). In the full-information case this can be seen from Table 3 which gives the distribution of X_τ for the case $\xi = B$ and $\xi = S$. Clearly, X_t tends to be in a higher state (compare the high probabilities for x_8) at a default. Hence we expect that the simplified value adjustments given in (2.5) underestimate the true value adjustments by a sizeable amount. This is indeed true, see the numbers reported in the last row of Table 7.

6.3. *Results for the case with collateralization*

We go on with the analysis of various collateralization strategies. Since collateralization is only relevant on paths where $\tau < T$ and where $\xi \in \{B, S\}$, we illustrate the performance of collateralization strategies by plotting the conditional distribution function of the random variables

$$
\begin{aligned}
L_B(C) &:= 1_{\{\xi=S\}}\big(\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+\big)\\
L_S(C) &:= D(t,\tau)1_{\{\xi=B\}}\big(\mathrm{LGD}_B(C_{\tau-}^- - P_\tau^-)^- + \mathrm{LGD}'_B(P_\tau^+)^- - C_{\tau-}^+\big)\,,
\end{aligned}
$$

given that $\{\tau \le T, \xi \in \{B, S\}\}$. Note that for a given collateralization strategy C, $L_B(C)$ gives discounted loss to B that arises from an early default of S, whereas $L_S(C)$ gives the discounted loss to S that arises from an early default of B. We analyzed strategies of the following type:

- Threshold-collateralization with initial margin γ and thresholds $M_1 = M_2 := M$, denoted $C^{\gamma,M}$;
- Market collateralization $C^{\text{market}} = C^{0,0}$;
- The strategy C^* derived in Proposition 5.1 and the "model-independent optimal strategy" based on the estimators (5.7) and (5.8) for the incomplete-information model.

Our findings can be summarized as follows:

a) Threshold collateralization with $\gamma = 0$ is very effective in the complete-information model. For a threshold $M > 0$ counterparty risk is largely reduced as can be seen from Table 7. Counterparty credit risk even vanishes completely for $M = 0$. Moreover, losses are bounded when threshold-collateralization is used. This can be seen from Figure 2 which displays the empirical cdf of L_B given $\tau \le T$ and $\xi \in \{B, S\}$ in the complete information model for different scenarios.

b) Under incomplete information the performance of threshold collateralization with $\gamma = 0$ and threshold M is not fully satisfactory. The main reason is the fact that because of the contagion effects threshold collateralization systematically underestimates the market value of the CDS at τ which leads to losses for the protection buyer in case that $\xi = S$. As a consequence we observe high values for the CCVA in scenarios such as the Base scenario where $\mathbb{Q}(\tau \le T, \xi = S)$ is comparatively high, compare Table 7. The losses of the protection seller on the other hand are always smaller than the threshold M. This behavior can be seen from Figures 3 and 4 where the conditional cdf of L_B and L_S is plotted in various scenarios.

A nonzero initial margin γ can improve the performance of threshold collateralization in scenarios where the credit quality of B is much better than the credit quality of S as in the Base scenario. In that case $\mathbb{Q}(\xi = S \mid \tau \le T, \xi \in \{B, S\})$ is close to one and one essentially knows that $\xi = S$ in case of an early default. Consequently it is possible to hedge a large part of the contagion effects by choosing a positive initial margin γ. This can be seen from Figure 5 where $m(C^{\gamma,M})$ is plotted in the Base scenario for various values of γ and M. In a symmetric scenario where B and S have similar credit quality on the other hand, the identity of the first defaulting firm cannot be predicted and choosing a nonzero initial margin does not help much to improve the effectiveness of threshold collateralization, as can be seen from Figure 6. This is clear intuitively: a large initial margin $\gamma > 0$ will lead to a loss for S in case that $\xi = B$ because of re-hypothecation; on the other hand for $\gamma \le 0$ there will be a loss for B in case that $\xi = S$ because of contagion effects, and neither of the two cases can be ruled out a-priori because B and S have similar credit quality.

c) The optimal strategy C^* from Proposition 5.1 on the other hand performs well under incomplete information and reduces counterparty risk substantially as can be seen from Table 8 where various credit value adjustments and the value of $m(C)$ are given. The strategy is particularly effective in scenarios where the credit quality of B is higher than the credit quality of S so that $x_S \le x_B$ such as the Base scenario and the Risky-PS scenario. On the other hand C^* does not fully eliminate counterparty risk in scenarios where the credit quality of S is worse than the credit quality of B such as the Base2 and the Risky-PB scenario, as is evident from Table 8. However, even in these scenarios the probability that some party suffers a large loss is fairly small. Of course, the superior performance of the refined collateralization strategy is related to the fact that within our model the quantities x_B and x_S can be computed exactly. We therefore compared the performance of C^* to the performance of the "model-independent optimal strategy" based on (5.7) and (5.8) on the one hand and to the performance of market-value collateralization on the other (see Table 8. Of course the model-independent version of our strategy performs worse than C^*. However, in scenarios where there is a non-negligible probability that the protection seller defaults first it performs significantly better than market-value collateralization, This shows that refined collateralization strategies that account for contagion effects have the potential to reduce counterparty credit risk significantly.

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Appendix A. Proofs

Proof. (Proof of Proposition 4.1) By symmetry, it suffices to consider the case $i = B$.
a) The default time τ_B is the time, at which the Markov chain (X, H^B) first enters the absorbing set $A = \{(1,1), \ldots, (K,1)\}$ and leaves the set $A^c := \{(1,0), \ldots, (K,0)\}$. Hence we get:

$$\begin{aligned}\mathbb{Q}\left(\tau_B > s \middle| X_t = k, H_t^B = 0\right) &= \mathbb{Q}\left((X_s, H_s^B) \in A^c \middle| X_t = k, H_t^B = 0\right) \\ &= 1_{\{\tau_B > t\}}(e_k^\top, 0)e^{Q(s-t)}(\mathbf{1}^{K^\top}, 0)^\top\end{aligned}$$

Here Q denotes the generator of (X, H^B). Q^n is of the form:

$$Q^n = \begin{pmatrix} W - \Lambda_B & \Lambda_B \\ 0 & W \end{pmatrix}^n = \begin{pmatrix} (W - \Lambda_B)^n & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} Q_B^n & * \\ 0 & * \end{pmatrix}.$$

Therefore the entries in the upper left part of the matrix exponential $e^{Q(s-t)}$ are given by $e^{Q_B(t-s)}$ and we can conclude:

$$\mathbb{Q}\left(\tau_B > s \middle| X_t = k, H_t^B = 0\right) = 1_{\{\tau_B > t\}} e_k^\top e^{Q_B(s-t)} \mathbf{1}^K.$$

b) The default times are conditionally independent doubly stochastic random times, and hence the first-to-default time exhibits an intensity which is given by the sum of the individual intensities (see McNeil, Frey and Embrechts (2005), Lemma 9.36), and the result follows from a).

c) We consider the Markov chain $\Psi_t = (X, H^B, H^R, H^S)_{\tau \wedge t}$ (the chain stopped at the first default time.) Ignoring the states where more than one company defaults (and which can therefore never be reached by Ψ), the infinitesimal generator of Ψ is given by:

$$\bar{Q} = \begin{pmatrix} W - \sum_{j \in \{B,R,S\}} \Lambda_j & \Lambda_B & \Lambda_R & \Lambda_S \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The protection buyer B defaults first and before time s if and only if the stopped Markov chain Ψ is in the set $\tilde{A} := \{(1,1,0,0), \ldots, (K,1,0,0)\}$ at time s. Therefore:

$$\begin{aligned}\mathbb{Q}\left(\tau \le s, \xi = B | X_t = k, H_t = (0,0,0)\right) &= \mathbb{Q}\left(\Psi_s \in \tilde{A} \middle| \Psi_t = (k,0,0,0)\right) \\ &= 1_{\{\tau > t\}}(e_k^\top, 0)e^{\bar{Q}(s-t)}(0, \mathbf{1}^{K^\top}, 0, 0)^\top.\end{aligned}$$

So we have to compute the entries of a submatrix of the matrix exponential $e^{\bar{Q}(s-t)}$. Since the n-th power of the matrix $\bar{Q}(s-t)$ is given by $(n > 0)$:

$$(\bar{Q}(s-t))^n = (s-t)^n \begin{pmatrix} Q_{(1)}^n & Q_{(1)}^{n-1}\Lambda_B & Q_{(1)}^{n-1}\Lambda_R & Q_{(1)}^{n-1}\Lambda_S \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the relevant submatrix is given by

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{Q_B^{n-1}}{n!}(s-t)^n \Lambda_B &= Q_{(1)}^{-1} \left(\sum_{n=0}^{\infty} \frac{Q_{(1)}^{n-1}}{n!}(s-t)^n - I \right) \Lambda_B \\ &= Q_{(1)}^{-1} \left(e^{Q_{(1)}(s-t)} - I \right) \Lambda_B, \end{aligned}$$

and the claim follows.

d) The result follows from similar considerations as in c). $\square$

Appendix B. Figures and Tables

Table 2. Results of model calibration for the base scenario

state	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
π_0	0.0810	0.0000	0.2831	0.0548	0.0000	0.0000	0.0000	0.5811
λ_B	0.0000	0.0010	0.0027	0.0040	0.0050	0.0059	0.0091	0.0195
λ_R	0.0031	0.0669	0.1187	0.1482	0.1687	0.1855	0.2393	0.3668
λ_S	0.0007	0.0245	0.0482	0.0627	0.0732	0.0818	0.1108	0.1840

Table 3. Distribution of X at τ in the base scenario for $\xi = B$ and $\xi = S$.

state	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$\xi = B$	0.0001	0.0144	0.0740	0.0500	0.0208	0.0221	0.0982	0.7203
$\xi = S$	0.0011	0.0309	0.1188	0.0713	0.0277	0.0279	0.1074	0.6149

Table 4. The first-to-default probabilities for different scenarios

scenario	B	R	S
Base	0.0293	0.4238	0.2463
Base2	0.2463	0.4238	0.0293
Symmetric	0.1851	0.3972	0.1851
RiskyPB	0.4238	0.2463	0.0293
RiskyPS	0.0293	0.2463	0.4263

Table 5. Value adjustments in different scenarios for the complete-information model (left) and for the incomplete information model (right), both for the uncollateralized case.

	full information			incomplete information		
scenario	CVA	DVA	BCVA	CVA	DVA	BCVA
Base	94	1	92	83	1	82
Base2	10	26	-16	9	15	-6
Symmetric	74	5	68	72	4	68
RiskyPB	6	45	-39	6	27	-21
RiskyPS	115	1	114	97	1	96

Table 6. Un-collateralized value adjustments under incomplete information for different values of the parameter c (low values of c correspond to a high observation noise) in the base scenario

noise parameter	CVA	DVA	BCVA
$c = 0$	68	0	68
$c = 1$	83	1	82
$c = 2$	89	1	88
$c = 5$	92	1	90

Table 7. Value adjustments in the complete-information model (left) and in the incomplete-information model (right) with threshold-collateralization and market value collateralization ($M_1 = M_2 = 0$) for $\gamma = 0$ in the Base scenario. In the last row we report the value adjustment corresponding to the simplified value adjustment formula (2.5).

	full information			incomplete information		
threshold	CCVA	CDVA	BCCVA	CCVA	CDVA	BCCVA
$M_1 = M_2 = 0$	0	0	0	35	0	35
$M_1 = M_2 = 0.02$	16	0	15	45	0	45
$M_1 = M_2 = 0.05$	38	1	37	60	0	60
no collateralization with						
(i) correct formula	93	1	92	83	1	82
(ii) simplified formula	68	6	62	54	4	49

Table 8. Performance of different collateralizion strategies in the incomplete-information model as measured by $m(C) = \text{CCVA} + \text{CDVA}$. Note that $m(C^*)$ is small in all scenarios and that $m(C^*) = 0$ in the Base- and RiskyPS scenarios where $x_S \leq x_B$. Moreover, the strategy based on the model-independent estimators (5.7) and (5.8) performs much better than market-value collateralization in all scenarios where there is a non-negligible probability that S defaults first.

scenario	C^*	C^{market}	C^* based on (5.7) and (5.8)
Base	2	36	24
Base2	5	7	7
Symmetric	2	32	22
RiskyPB	8	8	8
RiskyPS	0	41	27

Fig. 1. Trajectories of the fair CDS spread in the complete and incomplete information model.

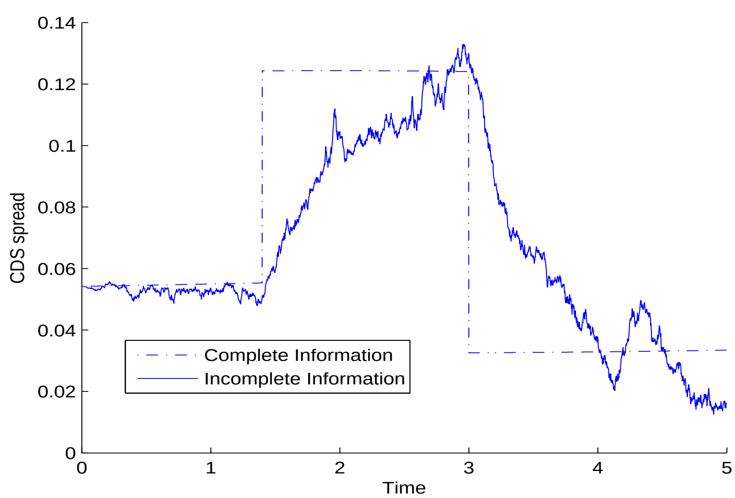

Fig. 2. Empirical cdf of L_B for different threshold-collateralization strategies with $\gamma = 0$ in the Base scenario in the complete-information model given $\tau \leq T$ and $\xi \in \{B, S\}$. Note that without collateralization the probability that L_B is large is quite high since in the base scenario $\mathbb{Q}(\xi = S \mid \tau \leq T, \xi \in \{B, S\}) = 0.245/(0.245 + 0.029) \approx 1$ (see Table 4). We can see that threshold collateralization reduces counterparty credit risk very effectively in that case.

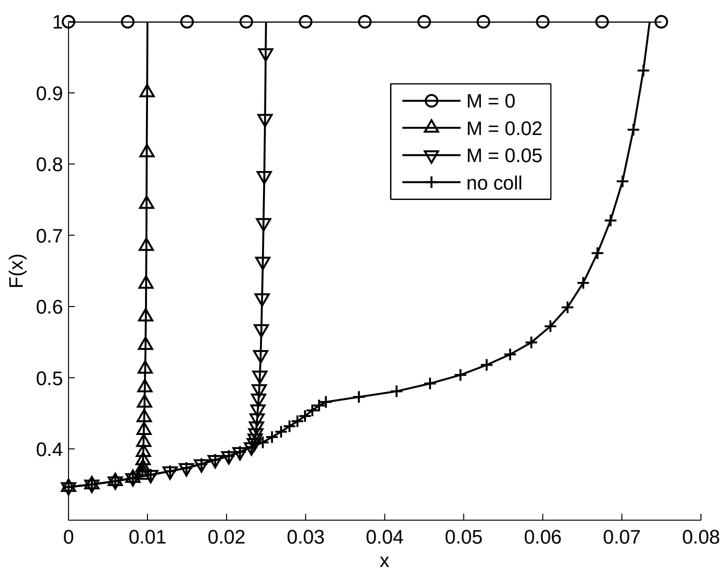

Fig. 3. Empirical cdf of L_B for different threshold-collateralization strategies with $\gamma = 0$ in the Base scenario in the incomplete-information model given $\tau \leq T$ and $\xi \in \{B, S\}$. In that case threshold collateralization with $\gamma = 0$ is not very effective: even for $M = 0$ there is roughly a 20% probability that L_B exceeds 300bp.

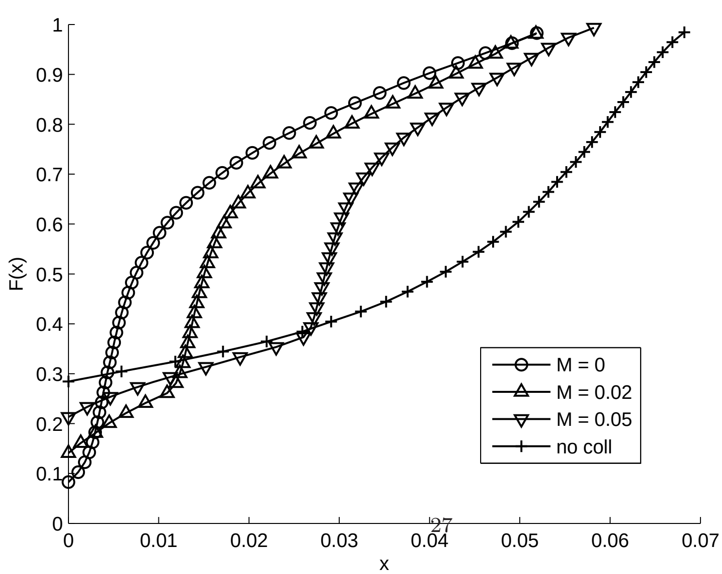

Fig. 4. Empirical cdf of L_S using threshold-collateralization for the Base2 scenario in the incomplete-information model given $\tau \leq T$ and $\xi \in \{B, S\}$. In this scenario $\mathbb{Q}(\xi = B \mid \tau \leq T, \xi \in \{B, S\})$ is close to one so that threshold collateralization is quite effective even under incomplete information.

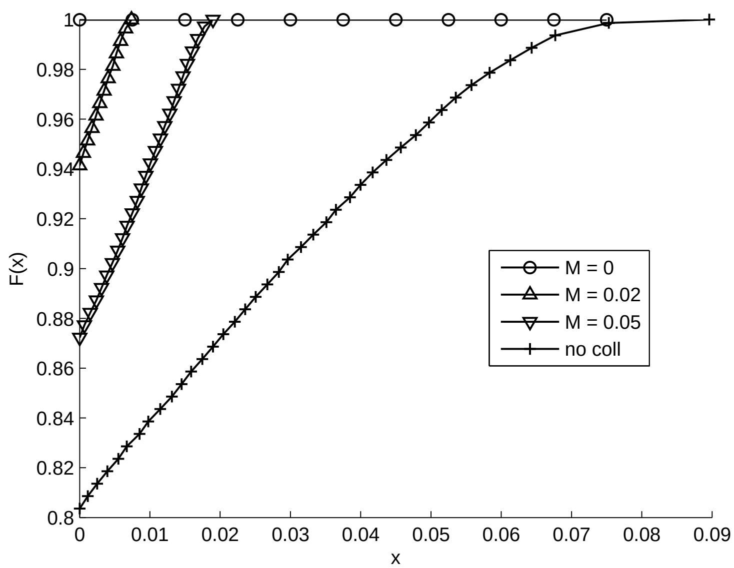

Fig. 5. Graph of $m(C^{\gamma,M})$ (sum of CCVA and CDVA) under incomplete information for the threshold strategy $C^{\gamma,M}$ for varying values of the initial margin γ and the threshold M in the Base scenario. The function $m(C^{\gamma,M})$ is minimal for $M = 0$ and a positive initial threshold $\gamma^* \approx 0.12$ leading to an optimal value $m(C^{\gamma^*,0}) = 3$bp, so that counterparty risk can in effect be mitigated by a proper choice of the initial margin.

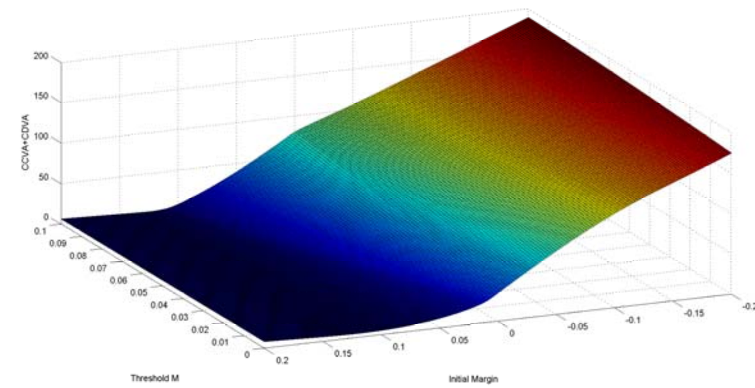

Fig. 6. Graph of $m(C^{\gamma,M})$ (sum of CCVA and CDVA) under incomplete information for the threshold strategy $C^{\gamma,M}$ for varying values of the initial margin γ and the threshold M in the symmetric scenario. The function $m(C^{\gamma,M})$ is minimal for $M = 0$ and a small initial threshold $\gamma^* \approx 0.01$. Note that $m(C^{\gamma^*,0}) = 15$bp whereas for the optimal strategy from Proposition 5.1 one has $m(C^*) = 0$.