Reserve Risk Dependencies

under Solvency II and IFRS 4 perspective

René Dahms

rene.dahms at math.ethz.ch

Abstract

In modern times, in particular under Solvency II, SST and IFRS 4 perspective, it is required to specify the uncertainty corresponding to the estimation of the expectation of the outstanding liabilities reserving. In order to do that often some assumptions about the distribution of the estimated losses are made and the actuaries estimate the corresponding parameters, for instance one assumes a Lognormal distribution and estimates the mean and the variance. For a single portfolio this has been studied by several authors. In this article we want to look at collections of portfolios, use Linear-Stochastic-Reserving-Methods (*LSRMs*) to couple them in an often natural way and estimate the covariance matrix of the corresponding reserve risk.

In Section 1 we will very briefly recall some statements about *LSRMs*. Section 2 will show how to estimate the reserving uncertainty in terms of a covariance matrix. We will do that for the ultimate (IFRS 4) as well as for the one year claims development (Solvency II or SST) point of view.

Finally we will present two examples in Section 3. Both are based on a sample implementation of *LSRMs* (under the license of GPL 3) that can be obtained from the author.

Keywords: Stochastic Reserving, Mean Squared Error of Prediction, Solvency Reserving Risk, Claims Development Result, IFRS4, Dependent portfolios.

1 Brief introduction to *LSRMs*

We want to look at a collection of portfolios, use Linear-Stochastic-Reserving-Methods (*LSRMs*), see [2], to estimate the total reserves and the corresponding uncertainty. Since *LSRMs* can handle payments, reported amounts, reported number of claims and other triangle based properties of a collection of portfolios simultaneously, we call the corresponding triangles claim properties. The main idea behind *LSRMs* is the assumption that the (conditionally) expected changes of claim properties during a development period are proportional to exposures which depend linearly on the past of those claim properties. Another important property of *LSRMs* is that they allow for various dependencies of accident periods and claim properties. Many of the classical reserving methods, like the Chain-Ladder-Method, the Complementary-Loss-Ratio-Method and the Bornhuetter-Ferguson-Method are *LSRMs*. A detailed discussion of *LSRMs* can be found in [2]. Note, the published version of this article contains typing error, which makes the implementation very hard. A corrected version can be obtained from the author. Moreover, a sample implementation of *LSRMs* is available under the license GPL 3.

In this section we want to repeat the basic notations and results about *LSRMs*.

Let $S^m_{i,k}$, $0 \leq m \leq M$, $0 \leq i \leq I$, $0 \leq k \leq J$, denote the incremental value of the m-th claim property of the i-th accident period during the k-th development period. We assume that $I \geq J$ and that there is no development of any claim property after development period J, which means we do not discuss any tail development. Such claim properties may be the usual candidates like payments, reported amounts and number of reported claims or even more special constructions like payments after reopening.

Our model contains three natural time lines: accident periods or rows, development periods or columns and business periods or lower-left to upper-right diagonals. We will use the indices i and h for accident periods, j and k for development periods, l and m for claim properties and n for business periods, see Figure 1.

By $\mathbb{L}^n$ and $\mathbb{L}_k$ we denote the linear spaces generated by all increments $S^m_{i,j}$ up to business period n and development period k, respectively. Moreover, by $\mathbb{L}^n_k$ we denote the linear space generated by $\mathbb{L}^n$ and $\mathbb{L}_k$, i.e.

$$
\begin{aligned}
\mathbb{L}^n &:= \left\{ \sum_{m=0}^{M} \sum_{i=0}^{I} \sum_{j=0}^{(n-i)\wedge J} x_{i,j}^m S_{i,j}^m \ : \ x_{i,j}^m \in \mathbb{R} \right\}, \\
\mathbb{L}_k &:= \left\{ \sum_{m=0}^{M} \sum_{i=0}^{I} \sum_{j=0}^{k} x_{i,j}^m S_{i,j}^m \ : \ x_{i,j}^m \in \mathbb{R} \right\}, \\
\mathbb{L}_k^n &:= \left\{ \sum_{m=0}^{M} \sum_{i=0}^{I} \sum_{j=0}^{((n-i)\wedge J)\vee k} x_{i,j}^m S_{i,j}^m \ : \ x_{i,j}^m \in \mathbb{R} \right\},
\end{aligned} \tag{1.1}
$$

where $a \wedge b$ and $a \vee b$ denote the minimum and maximum of the real numbers a and b, respectively. The σ-algebra of all information of accident period i up to development period k is denoted by $\mathcal{B}_{i,k}$. Moreover, we denote the σ-algebras generated by $\mathbb{L}^n$, $\mathbb{L}_k$ and $\mathbb{L}_k^n$ by $\mathcal{D}^n$, $\mathcal{D}_k$ and $\mathcal{D}_k^n$, respectively, i.e.

$$
\mathcal{B}_{i,k} := \sigma\left(S_{i,j}^m \ : \ 0 \le m \le M, \ 0 \le j \le k \right), \quad \mathcal{D}_k := \sigma\left(\mathbb{L}_k\right) = \sigma\left(\bigcup_{i=0}^{I} \mathcal{B}_{i,k} \right),
$$

$$
\mathcal{D}^n := \sigma\left(\mathbb{L}^n\right) = \sigma\left(\bigcup_{i=0}^{I} \mathcal{B}_{i,(n-i)\wedge J} \right), \qquad \mathcal{D}_k^n := \sigma\left(\mathbb{L}_k^n\right) = \sigma\left(\bigcup_{i=0}^{I} \mathcal{B}_{i,((n-i)\wedge J)\vee k} \right),
$$

see Figure 1. We call the information $\mathcal{D}_k^{i+k}$ the past of $S_{i,k+1}^m$, $0 \le m \le M$.

Figure 1: Claim property triangle

Assumption A *The stochastic model of the increments $S_{i,k}^m$ is a Linear-Stochastic-Reserving-Method (LSRM) if there exist constants f_k^m and $\sigma_k^{m_1,m_2}$ such that*

i) for all i, m and k the expectation of the claim property $S_{i,k+1}^m$ under the condition of all information of its past $\mathcal{D}_k^{i+k}$ is proportional to an exposure $R_{i,k}^m$

contained in $\mathbb{L}^{i+k} \cap \mathbb{L}_k$, i.e.

$$\mathrm{E}\left[S_{i,k+1}^m \middle| \mathcal{D}_k^{i+k}\right] = f_k^m R_{i,k}^m \in \mathbb{L}^{i+k} \cap \mathbb{L}_k. \tag{1.2}$$

ii) for all i, m_1, m_2 and k the covariance of the claim properties $S_{i,k+1}^{m_1}$ and $S_{i,k+1}^{m_2}$ under the condition of all information of their past $\mathcal{D}_k^{i+k}$ is proportional to an exposure $R_{i,k}^{m_1,m_2}$ contained in $\mathbb{L}^{i+k} \cap \mathbb{L}_k$, i.e.

$$\mathrm{Cov}\left[S_{i,k+1}^{m_1}, S_{i,k+1}^{m_2} \middle| \mathcal{D}_k^{i+k}\right] = \sigma_k^{m_1,m_2} R_{i,k}^{m_1,m_2} \in \mathbb{L}^{i+k} \cap \mathbb{L}_k. \tag{1.3}$$

Since the exposures $R_{i,k}^m$ and $R_{i,k}^{m_1,m_2}$ depend linearly on the past of the claim properties $S_{i,k+1}^m$ there exists exposure parameters $\gamma_{i,k,h,j}^{m,l}$ and $\gamma_{i,k,h,j}^{m_1,m_2,l}$ such that

$$R_{i,k}^m =: \sum_{l=0}^{M} \sum_{h=0}^{I} \sum_{j=0}^{(i+k-h)\wedge k} \gamma_{i,k,h,j}^{m,l} S_{h,j}^l \quad \text{and} \quad R_{i,k}^{m_1,m_2} =: \sum_{l=0}^{M} \sum_{h=0}^{I} \sum_{j=0}^{(i+k-h)\wedge k} \gamma_{i,k,h,j}^{m_1,m_2,l} S_{h,j}^l, \tag{1.4}$$

Remark 1.1 *Many of the standard reserving methods are LSRMs. For instance (we only specify the exposure parameters where they are not equal to zero):*

Chain-Ladder-Method, see [3]: *We have only one claim property and*

$$\gamma_{i,k,h,j}^{0,0} = \gamma_{i,k,h,j}^{0,0,0} = 1, \qquad \text{iff } i = h \text{ and } j \leq k.$$

Complementary-Loss-Ratio-Method, see [4]: *We have one real claim property $S_{i,k}^0$ and a known exposure $U_i =: S_{i,0}^1$ and take*

$$\gamma_{i,k,h,j}^{0,1} = \gamma_{i,k,h,j}^{0,0,1} = 1, \qquad \text{iff } i = h \text{ and } j = 0.$$

Extended-Complementary-Loss-Ratio-Method, see [1]: *We have two claim properties, reported amounts $S_{i,k}^0$ and payments $S_{i,k}^1$, and take the case reserves as exposure, i.e.*

$$\gamma_{i,k,h,j}^{m,l} = \gamma_{i,k,h,j}^{m_1,m_2,l} = (-1)^l, \qquad \text{iff } i = h \text{ and } j \leq k.$$

2 Reserve risk and correlation

In the article [2] a mixture of claim properties $\sum_{m=0}^{M} \sum_{i=0}^{I} \alpha_i^m \sum_{k=0}^{J-1} S_{i,k+1}^m$ has been analysed and (conditionally) unbiased estimators for the development parameters f_k^m, the covariance parameters $\sigma_k^{m_1,m_2}$ and the future development of claim properties $S_{i,k}^m$ have been derived. Moreover, the ultimate uncertainty as well as the one period uncertainty have been analysed. The first may become very important in the context of IFRS 4 and the latter already is very important in under Solvency II and SST perspective.

In order to get a distribution for the ultimate outcome or the development result actuaries often choose one out of a class of distributions, for instance Lognormal, by fitting the first two moments. Therefore, in the context of *LSRMs* we have two possible ways in order to do so:

1. Modelling the collection of all portfolios simultaneously, estimating the mean (the reserves) and the variance (the mean squared error of prediction) and fitting one distribution.
2. Modelling each portfolio stand alone, estimating the corresponding means and a suitable covariance matrix in order to couple the fitted distributions.

Usually we would suggest the first approach. But sometimes you do not want or you are not allowed to do so. For that case we now want to show how *LSRMs* can be used in order to estimate a suitable covariance matrices for the ultimate uncertainty as well as for the one period uncertainty.

2.1 Ultimate uncertainty

In [2] the ultimate uncertainty has been analysed in terms of the mean squared error of prediction of the ultimate outcome:

$$\begin{aligned}
\operatorname{mse}\left[\sum_{m=0}^{M}\sum_{i=0}^{I}\alpha^m\alpha_i\sum_{k=0}^{J-1}\widehat{S}^m_{i,k+1}\right] &:= \mathrm{E}\left[\left(\sum_{m=0}^{M}\sum_{i=0}^{I}\alpha^m\alpha_i\sum_{k=0}^{J-1}\widehat{S}^m_{i,k+1}-S^m_{i,k+1}\right)^2\middle|\mathcal{D}^I\right]\\
&= \sum_{m_1,m_2=0}^{M}\alpha^{m_1}\alpha^{m_2}\beta^{m_1,m_2}, \qquad (2.1)
\end{aligned}$$

where α^m, α_i are arbitrary real numbers and

$$\begin{aligned}
&\beta^{m_1,m_2}\\
&:= \mathrm{E}\left[\left(\sum_{i_1=0}^{I}\alpha_{i_1}\sum_{k_1=I-i_1}^{J-1}\widehat{S}^{m_1}_{i_1,k_1+1}-S^{m_1}_{i_1,k_1+1}\right)\left(\sum_{i_2=0}^{I}\alpha_{i_2}\sum_{k_2=I-i_2}^{J-1}\widehat{S}^{m_2}_{i_2,k_2+1}-S^{m_2}_{i_2,k_2+1}\right)\middle|\mathcal{D}^I\right]\\
&\approx \mathrm{Cov}\left[\left(\sum_{i_1=0}^{I}\alpha_{i_1}\sum_{k_1=I-i_1}^{J-1}\widehat{S}^{m_1}_{i_1,k_1+1}-S^{m_1}_{i_1,k_1+1}\right),\left(\sum_{i_2=0}^{I}\alpha_{i_2}\sum_{k_2=I-i_2}^{J-1}\widehat{S}^{m_2}_{i_2,k_2+1}-S^{m_2}_{i_2,k_2+1}\right)\middle|\mathcal{D}^I\right].
\end{aligned}$$

The approximation in the last line is due to the fact that $\sum_{i=0}^{I} \alpha_i \sum_{k_1=I-i}^{J-1} \widehat{S}_{i,k+1}^{m}$ is only an unbiased but not a $\mathcal{D}^I$-conditionally unbiased estimator.

Since the mixing weights α^m in (2.1) are arbitrary we can interpret the matrix $(\beta^{m_1,m_2})_{0\le m_1,m_2\le M}$ as covariance matrix of the ultimate reserve risk of the collection of claim properties. Since the estimators of the ultimate uncertainty, presented in [2], are of the same form we are done.

Remark 2.1

1. *We can use the mixing weights α^m and α_i in order to get the right covariance matrix. For instance if you look at a collection of payment (or reported amount) triangles of different portfolios then with $\alpha_i \equiv 1$ the equation* (2.1) *gives you directly the corresponding covariance matrix. But if you look at a collection of payment and reported amount triangles of different portfolios then you have to specify how much credibility you will assign to the projection of payments and the projection of reported amounts of each portfolio, i.e. choose α^m and α_i accordantly and subsume in* (2.1)*.*
2. *Although the matrix $(\beta^{m_1,m_2})_{0\le m_1,m_2\le M}$ is by definition always covariance matrix, that mean it is always a symmetric positive semidefinite matrix, the corresponding estimate may not be positive semidefinite. In particular if at least one eigenvalues of $(\beta^{m_1,m_2})_{0\le m_1,m_2\le M}$ is almost or equal to zero the corresponding eigenvalue of the estimated covariance matrix may be slightly negative. Therefore, you always have to check the estimated results and apply actuarial judgement if necessary.*

2.2 Solvency uncertainty

Under the solvency uncertainty one usually understands the uncertainty of the development result

$$\sum_{m=0}^{M}\sum_{i=0}^{I}\sum_{k=I-i}^{J-1} \widehat{S}_{i,k+1}^{m,I+1} - \widehat{S}_{i,k+1}^{m,I}$$

of the next business period $I+1$. Here the additional upper index specifies the time of estimation.

Like in the case of the ultimate uncertainty the uncertainty of the development result has been analysed in [2] in terms of the mean squared error of prediction

$$\begin{aligned}
&\operatorname{mse}\left[\sum_{m=0}^{M}\sum_{i=0}^{I}\alpha^m\alpha_i\sum_{k=0}^{J-1}\widehat{S}_{i,k+1}^{m,I+1}-\widehat{S}_{i,k+1}^{m,I}\right] \qquad (2.2)\\
&:= \mathrm{E}\left[\left(\sum_{m=0}^{M}\sum_{i=0}^{I}\alpha^m\alpha_i\sum_{k=0}^{J-1}\widehat{S}_{i,k+1}^{m,I+1}-\widehat{S}_{i,k+1}^{m,I}\right)^2\middle|\mathcal{D}^I\right]=\sum_{m_1,m_2=0}^{M}\alpha^{m_1}\alpha^{m_2}\delta^{m_1,m_2},
\end{aligned}$$

where

$$\begin{aligned}
&\delta^{m_1,m_2}\\
&:= \mathrm{E}\left[\left(\sum_{i_1=0}^{I}\alpha_{i_1}\sum_{k_1=I-i_1}^{J-1}\widehat{S}_{i_1,k_1+1}^{m_1,I+1}-\widehat{S}_{i_1,k_1+1}^{m_1,I}\right)\left(\sum_{i_2=0}^{I}\alpha_{i_2}\sum_{k_2=I-i_2}^{J-1}\widehat{S}_{i_2,k_2+1}^{m_2,I+1}-\widehat{S}_{i_2,k_2+1}^{m_2,I}\right)\middle|\mathcal{D}^I\right]\\
&\approx \operatorname{Cov}\left[\left(\sum_{i_1=0}^{I}\alpha_{i_1}\sum_{k_1=I-i_1}^{J-1}\widehat{S}_{i_1,k_1+1}^{m_1,I+1}-\widehat{S}_{i_1,k_1+1}^{m_1,I}\right),\left(\sum_{i_2=0}^{I}\alpha_{i_2}\sum_{k_2=I-i_2}^{J-1}\widehat{S}_{i_2,k_2+1}^{m_2,I+1}-\widehat{S}_{i_2,k_2+1}^{m_2,I}\right)\middle|\mathcal{D}^I\right].
\end{aligned}$$

The approximation in the last line is due to the fact that the expectation but not the $\mathcal{D}^I$-conditioned expectation of $\sum_{i=0}^{I}\alpha_i\sum_{k_1=I-i}^{J-1}\widehat{S}_{i,k+1}^{m,I+1}-\widehat{S}_{i,k+1}^{m,I}$ equals zero. Now we can argue in the same way a in Section 2.1 in order to get estimates for the corresponding covariance matrix.

3 Examples

References

[1] Dahms, René (2008). A loss reserving method for incomplete data. SAV Bulletin, Vol. 1&2, 127–148.

[2] Dahms, Renè (2012). Linear Stochastic Reserving Methods. ASTIN Bulletin, Vol. 42/1, 1–34.

[3] Mack, Thomas (1993). Distribution-free calculation of the standard error of chain ladder reserve estimates. ASTIN Bulletin, Vol. 23/2, 213–225.

[4] Mack, Thomas (1997). Schadenversicherungsmathematik. Karlsruhe, Verlag Versicherungswirtschaft.