Stochastic Models in Credit Portfolio Management

CRM 2001- Vienna

Ludger Overbeck

Risk Research & Development

Group Risk Management

Deutsche Bank AG

Frankfurt, Germany

Overview

- Multiple obligors (upto 100.000 or more)

- Losses caused by defaults

Two main models

- Static one period model, i.e. only defaults in one year.

Risk Management

Risk Capital Allocation

Pricing/Valuation of Collaterized Loan/Bond Obligations, ABS/CMBS and their first loss positions.

- Modeling the default times, Pricing of Basket Credit Derivatives and also CLO structures and tranches.

Basic Model

Random variable describing defaults:

$$L = \sum_{i=1}^{m} l_i \mathbf{1}_{D_i}$$

$$\begin{aligned} m &= \text{\# of counterparties} \\ D_i &= \text{Default of counterparty} \\ l_i &= \text{Loss in the event of default} \\ &= \text{Exposure * Loss Given Default} \end{aligned}$$

The exposure is viewed as a loan equivalent exposure, if the transaction with a counterparty is a traded product (Average Expected Exposure)

Random Variable if credit migration is also considered:

$$L = \sum_{i=1}^{m} \sum_{r=1}^{k} l_i^r \mathbf{1}_{D_i^r}$$

k is the number of rating classes and l_i^r is the loss if counterparty i migrates from is rating class today $r(i)$ to rating class r. Assuming a continuous rating $r \in R$ the general formula is

$$L = \sum_{i=1}^{m} l_i^{r_i(\omega)},$$

where $r_i(\omega)$ is the random rating of counterparty i at time 1.

More general for each time t, we have

$$L_t(\omega) = \sum_{i=1}^{m} l_i(t, \omega, r_{t,i}(\omega))$$

Here $r_{t,i}(\omega)$ is the rating of counterparty i at time t and $l_i(t, \omega, r)$ is the loss of counterparty i at time t if i is in rating r.

Joint Default Probabilities

The determination of the probability of joint defaults of each pair of obligors is important. Single Defaults are derived from relative frequency of defaults in uniform segments.

$$P[D_i] \;=\; \frac{\text{\# Defaulted members in Segment A}}{\text{All members in Segment A}}$$

Joint Defaults?

$$P[\text{CP in Seg A defaults}, \text{CP in Seg B defaults}]$$
$$= \frac{\#?}{\#???}$$

For joint defaults a model is needed!

Multivariate Ability to Pay

$(A_1^{(i)})_{i=1,..,m}$ =" Ability to pay" at year one of the vector of obligors.

Default in one year is then be defined by the event

$$D_i \ = \ \{A_1^{(i)} < C^{(i)}\},$$

where $C^{(i)}$ is a calibrated "Default Point".

<u>Remark</u>: If the counterparty is an exchange traded firm, then $A_t^{(i)}$ =Value of firm's assets= $F(E, L, t)$ where

$$\begin{aligned} E &= (E_t)_{t \geq 0} \text{ value of equities} \\ L &= (L_t)_{t \leq 0} \text{ Liability Process .} \end{aligned}$$

The function F may be derived by modeling the equity as a contingent claim on the firm (Call option as in Merton '74) or vice versa. "Default Point"= Function of Liabilities.

Joint Defaults if A is normal

$$
\begin{aligned}
& \mathsf{JDP}_{ij} \\
:= \; & P\left[A_1^{(i)} < C^{(i)}, A_1^{(j)} < C^{(j)}\right] \\
= \; & \int_{-\infty}^{N^{-1}(P[D_i])} \int_{-\infty}^{N^{-1}(P[D_j])} \frac{1}{2\pi\sqrt{(1-r_{ij}^2)}} \\
& \exp\left(\frac{1}{2(1-r_{ij})}(x_1^2 + x_2^2 - 2r_{ij}x_1x_2)\right)dx_1dx_2
\end{aligned}
$$

Default correlation

$$
\begin{aligned}
\rho_{ij} &= \mathsf{corr}(\mathbf{1}_{\{A_1^{(i)}<C^{(i)}\}}, \mathbf{1}_{\{A_1^{(j)}<C^{(j)}\}}) \\
&= \frac{\mathsf{JDP}_{ij} - P[D_i]P[D_j]}{\sqrt{P[D_i](1-P[D_i])P[D_j](1-P[D_j])}}
\end{aligned}
$$

Loss Distribution

Uniform Portfolio

$p_i = p, l_i = 1 \ \forall \ i = 1, .., m$ and $r_{ij} = r \ \forall \ i, j = 1, .., m, \ i \neq j$. Zerlege

$$W_t^i = \sqrt{r} B_t^0 + \sqrt{1-r} B_t^i,$$

where $B^j, j = 0, .., m$ are independent Brownian Motions.

Conditioning on the systematic factor B_1^0 yields for the percentage portfolio loss: $P[L = \frac{k}{m}] =$

$$\binom{m}{k} P\left[A_1^1 < C_1, .., A_1^k < C_k,\right.$$

$$\left. A_1^{k+1} > C_{k+1}, .., A_1^m > C_m\right]$$

$$= \binom{m}{k} \int_{-\infty}^{\infty} P\left[A_1^i < C_i, i = 1, .., k,\right.$$

$$\left. A_1^j > C_j, j = k+1, .., m \,\middle|\, B_1^0 = x\right] P[B_1^0 \in dx]$$

$$= \binom{m}{k} \int P\left[B_1^i < \frac{\ln \frac{C_i}{A_0^i} - (\mu_i - \frac{1}{2}\sigma_i^2) - \sigma_i \sqrt{r} x}{\sqrt{(1-r)} \cdot \sigma_i},\right.$$

$$i = 1, .., k,$$

$$\left. B_1^i > \frac{\ln \frac{C_i}{A_0^i} - (\mu_i - \frac{1}{2}\sigma_i^2) - \sigma_i \sqrt{\rho} x}{\sqrt{(1-r)} \sigma_i}, i = k+1, .., m\right]$$

$$P[B_1^0 \in dx]$$

$$= \binom{m}{k} \int_{-\infty}^{\infty} \Phi\left(-\frac{1}{\sqrt{1-r}}(c + \sqrt{r}x)\right)^k$$

$$\cdot \left(1 - \Phi\left(-\frac{1}{\sqrt{1-r}}(c + \sqrt{\rho}x)\right)\right)^{m-k} \Phi(dx).$$

In the last equation

$$c = c_i = \frac{1}{\sigma_i}(\ln(A_0^i/C_i) + \mu_i - \frac{1}{2}\sigma_i^2),$$

since $c_i = c = \Phi^{-1}(p)$.

Limiting distribution $m \to \infty$

$$\begin{aligned}
& F_m(\theta) \\
= \; & P[\text{percentage loss} < \theta] \\
= \; & \sum_{k=0}^{[m\theta]} P[\text{percentage loss} = k/m] \\
= \; & \sum_{k=0}^{[m\theta]} \binom{m}{k} \int_{-\infty}^{\infty} \Phi\left(-\frac{1}{\sqrt{1-r}}(c+\sqrt{r}x)\right)^k \\
& \quad \left(1 - \Phi\left(-\frac{1}{\sqrt{1-r}}(c+\sqrt{r}x)\right)\right)^{m-k} \Phi(dx)
\end{aligned}$$

Substitution

$$s = s(x) = \Phi\left(\frac{1}{\sqrt{1-r}} \cdot \left(\Phi^{-1}(p) - \sqrt{r}\cdot x\right)\right)$$

yields

$$\begin{aligned} F_m(\theta) &= \sum_{k=0}^{[m\theta]} \binom{m}{k} \int_0^1 s^k (1-s)^{m-k} \\ &\qquad d\Phi\left(\frac{1}{\sqrt{r}} \cdot (\sqrt{1-r}\cdot\Phi^{-1}(s) - \Phi^{-1}(p))\right) \end{aligned}$$

Because of the law of large numbers

$$\sum_{k=0}^{[m\theta]} \binom{m}{k} s^k (1-s)^{m-k} \to \mathbf{1}_{(0,\theta]}(s), 0 < s < 1$$

we obtain the density of the limiting distribution

$$\begin{aligned} \frac{\sqrt{1-r}}{\sqrt{r}} \exp\Big[-\frac{1}{2r} \cdot (\sqrt{1-r}\cdot\Phi^{-1}(s) \\ -\Phi^{-1}(p))^2 + \frac{1}{2}(\Phi^{-1}(s))^2 \Big]. \end{aligned}$$

Applications

Basket Credit Derivatives / Synthetic CDO's

Basic Concepts: "Sell" the risk of a subportfolio to the investors.

In mathematical terms: The holder of a tranch, that covers the losses between e.g. $\alpha\%$ and $\beta\%$ might have - depending on the contract specification - the (percentage) expected loss ($\sim$ spread)

$$s = \int \frac{(x-\alpha)^{+} \wedge (\beta-\alpha)}{(\beta-\alpha)} f_{p,\rho}(x)dx.$$

Since the overall expected loss (in percentage) is p we can obtain ρ.

Extending the above argument to portfolio consisting of many uniform portfolios we might try to derive implied correlations from a set of equations

$$s_i = \int \frac{(x-\alpha_i)^+ \wedge (\alpha_{i+1}-\alpha_i)}{(\alpha_{i+1}-\alpha_i)} f_{p_1,..,p_k,\rho_1,..,\rho_q}(x)dx,$$

$i = 1, .., l$. Here s_i is the spread of tranch i with boundaries α_i, α_{i+1}.

Economic Capital

Economic Capital is usually defined to be a quantile of the distribution of L minus the mean of L.

$$EC(\alpha) \;=\; q_\alpha(L) - E[L].$$

Assuming that the returns on a single asset in the portfolio are joint normal distributed, the portfolio return L is also normal. Then the economic capital is also given by multiples of the standard deviation. These multipliers CM are also called "Capital Multipliers".

$$EC(\alpha) \;=\; \sigma(L) * CM(\alpha).$$

Alternative Capital Definition

In light of the non-normality another Capital Definition should be considered. Economic Capital viewed as a Risk Measures should also satisfy the *Coherency Axioms* formulated by Artzner et al. A prominent example of a coherent risk measure is similar to a kind of lower partial moment

$$C(L) \;:=\; E[L|L > K],$$

where K is a threshold, used to define "Large Losses", e.g.
$K = q_{\alpha'}(L)$, then $C(L)$ is coherent.
$K =$fraction of equity capital
$K =$ experienced large losses

Contributory Economic Capital

This capital definition yields also a new definition of contributory economic capital

$$C_i(L) \;:=\; E[\mathbf{1}_{D_i} | L > K].$$

Average contribution of counterparty i to the portfolio loss, when large losses occur.

Theorem:

- $C(L) = \sum_{i=1}^{m} l_i C_i$

- If $K \notin \{\sum_{k=1}^{n} l_{i_k} | \{i_1, .., i_n\} \subset \{1, .., m\}\}$ then
$$C_i \;=\; \frac{\partial C(L)}{\partial^+ l_i}$$

- C_i is a coherent risk measure on the probability space generated by the portfolio.

Remarks

- $C_i < 1$

- These figures are a by-product if the loss distribution is generated by a Monte-Carlo-Simulation.

- First statistics of the distribution of L_i given $L > K$. Other statistics like variance could be useful. (Conditional variance is probably not coherent.)

- Definition reflects a causality relation. If counterparty i adds more to the overall loss than counterparty j in bad situations for the bank, also business with i should be more costly (asuming stand alone risk characteristics are the same).

- A function C is a coherent risk measure iff C is a generalized scenario, i.e. there is a set of probability measures $\mathcal{Q}$ such that

$$C(X) = sup\{E^{Q}[X] | Q \in \mathcal{Q}\}.$$

Since $L, L_i \geq 0$ the capital allocation rule carries over to all coherent risk measures.

Simulation Study

Portfolio of 40 counterparties with 5 year default probabilities. New capital allocation rule based on shortfall risk changes the order of capital consumption.

In the classical approach the contributory economic capital for transaction i is defined by

$$\gamma_i = (q_\alpha(L) - E[L]) \frac{\text{cov}(l_i \mathbf{1}_{D_i}, L)}{\sigma^2(L)}$$

Default Times

Given a default curve $p_i(t), t > 0$ for each obligor, where p_i is a strictly increasing function $(0, \infty) \to [0, 1]$.
Associated random variable $T^{(i)}$ is the default time of obligor i:

$$P[T^{(i)} \leq t] = p_i(t).$$

Goals: Construct $T^{(i)}$ as a simple first hitting time of a process Y, the ability to pay process.

Condition (*)

$$\exists Y^{(i)} = (Y_s^{(i)})_{s \geq 0}, C \in \mathbf{R},$$
$$\text{s.t. with } T_{C,Y^{(i)}} := \inf\{s | Y_s^{(i)} \leq C\}$$
$$P[T_{C,Y^{(i)}} \leq t] = p_i(t)$$

Correlation

Condition (**): Given a set of one year joint default probabilities $(p_{ij})_{i,j=1,..,m}$ we have

$$P[T_{C,Y^{(i)}} \leq 1, T_{C,Y^{(j)}} \leq 1] = p_{ij},$$
$$\forall i, j = 1, .., m, i \neq j.$$

Ansatz: Try to find a transformation $Y = F(W)$ of a Brownian motion W with correlation matrix $(\rho_{ij})_{i,j=1,..,m}$ such that (*) and (**) are satisfied.

Theorem. *There exist a correlation matrix* R *and a vector of time changes* T_t^i *such that* Y *defined by* $Y_t^i = W^i_{T_t}$, *where* W *is a Brownian motion with covariance matrix* R, *satisfies conditions (*) and (**).*

Proof: For a one-dimensional Brownian motion B the function

$$P[\inf_{s\leq t} B_s \leq C] \;=\; \tilde{f}(t, C)$$

is explicitly known, cf. Karatzas/Shreve:

$$\tilde{f}(t, C) \;=\; 2N(C/\sqrt{t}).$$

Therefore for each $Y_t^i = W_{T_t^i}$ with a deterministic time change T_t we have

$$P[\inf_{s\leq t} Y_s^i \leq C] \;=\; \tilde{f}(T_t^i, C).$$

Hence a time change defined by

$$\left(\frac{N^{-1}(p_i(t)/2)}{b}\right)^{-2} \;=:\; T_t^i$$

yields condition (*) for each i.

For a given correlation ρ_{ij} the joint default probability of two time changed Brownian motions equals

$$\begin{aligned} & P[T_{Y^i,C} \geq 1, T_{Y^i,C} \geq 1] \\ = & \int_C^\infty \int_C^\infty \breve{f}\left(x_1, x_2, \rho_{ij}, \min(T_1^i, T_1^j)\right) \\ & \left(1 - 2N\left(\frac{C - x_2}{\sqrt{\Delta}}\right)\right) dx_1 dx_2, \end{aligned}$$

where $\breve{f}(x_1, x_2, \rho_{ij}, t)$ is the corresponding density without time change at time t (as in the paper of Zhou on default correlation) and $\Delta = \max(T_1^i, T_1^j) - \min(T_1^i, T_1^j)$. $\diamond$

Remarks

- A transformation of Brownian motion based on deterministic time dependent volatility $Y = \int \sigma dW$ can also be written as a time change of a Brownian motion B. Then condition (*) can be met, however the correlation structure of B and W are different. Therefore the solution to condition (**) seems open.

- Open Problem:

  Find a non-random drift $g^{(i)}$ such that with the definition

  $$Y_t^i = W_t^i - \int_0^t g_s^i ds$$

  conditions (*) and (**) can be met.

Estimation of Correlation

- If Ability-to-Pay=Asset-Value and Asset-Value can be derived from equity time series and balance sheet information, the correlation can be obtained from time series.

- These firms provide factor model where non-listed firms can be embedded. Usually factor describing the systematic risk in the Ability-to-Pay is derived from balance sheet information (Sectors in which the company generates profits). It remains to determine the R-squared of the firms specific systematic factor.

- Large uniform retail portfolios

  As above, p=average default probability, ρ average asset correlation:

- General Approach:

  Use size of portfolios and consider losses $L_i, i = 1, ..., m$ in large subportfolios $\mathcal{S}_i$. Try to minimize KS-statistic of

  $$\left(F^{\rho}_{\widehat{L}_1,..,\widehat{L}_i}(\widehat{L}_i)\right)_{i=1,..,m}$$

  or other statistics of the conditional distribution of losses in portfolio i given the losses in other portfolios.

  Not yet implemented!

Validation of credit risk models

Default probabilities

$$P[D_i] = \text{Default Probabilities}$$

Determination

1. Step Rating

e.g. from 1="AAA", best creditworthiness to 10="C" worst

2. Step Calibration

$$P[D_i] = \frac{\#\text{Defaults in Rating } j(i)}{\#\text{in Rating } j(i)}$$

Challenge: Validation of default probability, usually they are assumed to be independent. But there are *Dependent defaults*.

Confidence bound for estimator depends on correlation!

But final objective is validation of EC-quota!

Validation

Is the EC quota correct? If the confidence level equals 99-98%, EC is only breeched in 1 out of 5000 years. You can't test this statistically!

Possible Approaches

1. Analysis in many subportfolios, i.e. Cross-Sectional Data instead of time series

Problem: Subportfolios are correlated

Try to identify portfolios which are almost uncorrelated

More ideas.

Randomized Subportfolios to make them independent?

2. Parametric Bootstrap.

Generate under H_0 many realisations of the "spatial" distribution of losses.

- Are these realisations "in the neighborhood" to the observed one?

- "Near" in the sense of point process distributions?

Statistical Tests about rejection of H_0, error probabilities

3. Parameter optimization (especially implied correlations), model selection, model validation with (non-parametric) boostrap techniques or resampling of dependent data?

Literature:

Efron/Tibshirani Chapter 17: Cross-Validation and other estimates of prediction error.

Davison/Hinkley Bootstrap Methods and their Application

Chapter 8, Complex Dependence (incl. Spatial processes).