Computing Optimal Investment Strategies Under Partial Information and Bounded Shortfall Risk

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Workshop and Mid-Term Conference on Advanced Mathematical Methods for Finance

Vienna, September, 17 – 22, 2007

Dynamic Portfolio Optimization

Financial market	containing risky and risk-free assets
	continuously tradable
	partial information on the drift
Initial capital	$x_0 > 0$
Horizon	$[0, T]$
Aim	**maximize expected utility** of terminal wealth
	constrain the risk of falling short a benchmark
Problem	find an **optimal investment strategy**

Financial Market Model

$(\Omega, \mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ filtered probability space

Stock market $dS_t = \mathrm{Diag}(S_t)\,(\,\mu_t\,dt + \sigma_t\,dW_t\,)$

with drift $\mu_t \in \mathbb{R}^n$ and volatility $\sigma_t \in \mathbb{R}^{n\times n}$

$dR_t = \left(\frac{dS_t^1}{S_t^1}, \dots, \frac{dS_t^n}{S_t^n}\right)^\tau$ return process

Money market risk-free interest rate r_t

(for simplicity we set $r_t \equiv 0$ and $\sigma_t \equiv \sigma$)

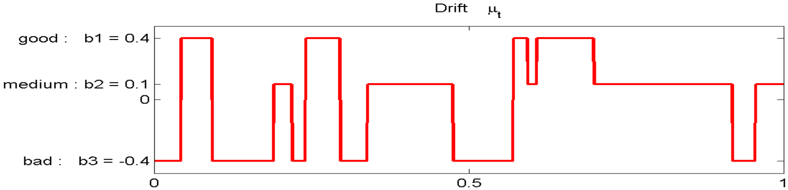
Martingale density $Z_t = \exp\left(-\int\limits_0^t \kappa_s^\tau\,dW_s - \frac{1}{2}\int\limits_0^t \|\kappa_s\|^2\,ds\right)$

with $\kappa_t = \sigma^{-1}\mu_t$ market price of risk

Martingale measure $\widetilde{P}(A) = E\,[Z_T\,1_A]$ for $A \in \mathcal{F}_T$

$\widetilde{P} \sim P$ and $\widetilde{W}_t = W_t + \int\limits_0^t \kappa_s\,ds$ is BM w.r.t. $\widetilde{P}$

Hidden Markov Model



Returns	dR_t	$=$	$\mu(Y_t)\,dt + \sigma\,dW_t$	observations
Drift	$\mu(Y_t)$	$=$	$B\,Y_t$	non-observable (hidden) state not $\mathcal{F}^S$-adapted

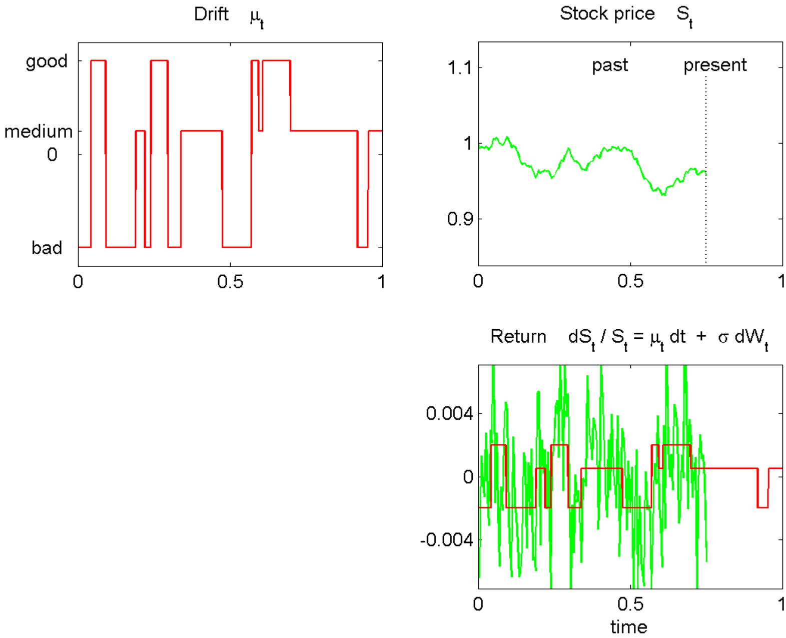
with Y_t time-continuous homogeneous Markov chain

independent of W_t

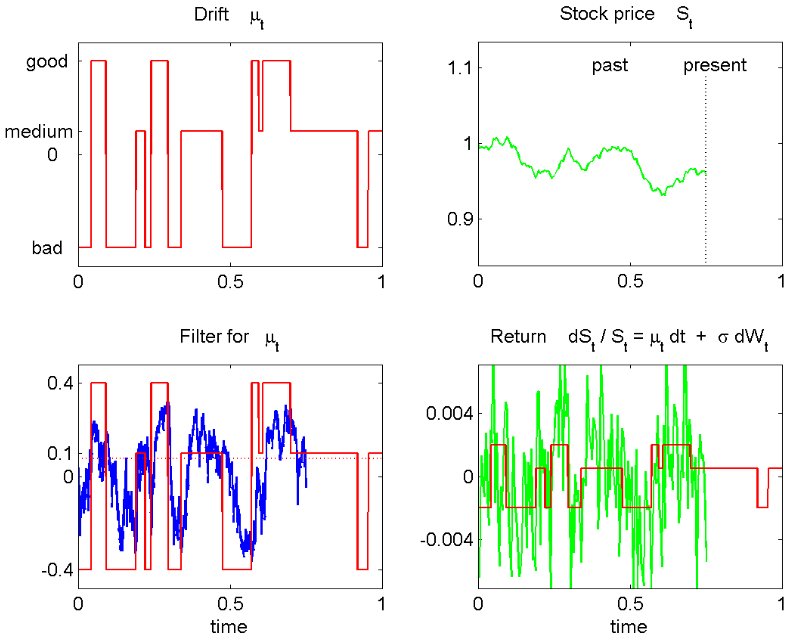
states of Y_t are unit vectors in $\mathbb{R}^d$: $e_1, \dots, e_d$

$n \times d$-matrix $B = (b^1, \dots, b^d)$, columns are states of μ_t

switching between the states is controlled by intensity matrix G



Drift μ_t
good
medium
0
bad
0
0.5
1
Stock price S_t
past
present
1.1
1
0.9
0
0.5
1
Return $dS_t / S_t = \mu_t\, dt + \sigma\, dW_t$
0.004
0
-0.004
0
0.5
1
time


Drift μt
good
medium
0
bad
0
0.5
1
Stock price St
1.1
1
0.9
past
present
0
0.5
1
Filter for μt
0.4
0.1
0
-0.4
0
0.5
1
time
Return dSt / St = μt dt + σ dWt
0.004
0
-0.004
0
0.5
1
time

Filter

Given observations of prices S_u (returns R_u) for $u \in [0, t]$

To find filter for state Y_t: $\eta_t = E\left[Y_t | \mathcal{F}_t^S\right] \quad \Rightarrow E\left[\mu_t | \mathcal{F}_t^S\right] = B\eta_t$

martingale density Z_t: $\zeta_t = E\left[Z_t | \mathcal{F}_t^S\right]$

Solution Wonham (1965), Elliot (1993)

unnormalized filter for state Y_t: $\mathcal{E}_t := \widetilde{E}\left[Z_T^{-1} Y_t | \mathcal{F}_t^S\right]$

$(\mathcal{E}_t)$ satisfies d-dimensional linear SDE

$$d\mathcal{E}_t = G^\tau \mathcal{E}_t\, dt + \mathrm{Diag}(\mathcal{E}_t) B^\tau (\sigma\sigma^\tau)^{-1} \underbrace{dR_t}_{\text{observations}}, \quad \mathcal{E}_0 = E\left[Y_0\right]$$

Filter for Z_t: $\zeta_t = (1_d^\tau \mathcal{E}_T)^{-1} = \frac{1}{\mathcal{E}_t^1 + \ldots + \mathcal{E}_t^d}$

for Y_t: $\eta_t = \zeta_t \mathcal{E}_t$

Portfolio

Initial capital $X_0 = x_0 > 0$

Wealth at time t $X_t = \underbrace{\pi_t^0}_{\text{bond}} + \underbrace{\pi_t^1}_{\text{stock 1}} + \ldots + \underbrace{\pi_t^n}_{\text{stock n}}$

invested in

Strategy $\boldsymbol{\pi_t = (\pi_t^1, \ldots, \pi_t^n)^\tau}$

Self financing condition $\Rightarrow$

Wealth equation

$$\begin{aligned} dX_t^\pi &= \boldsymbol{\pi_t^\tau}\,(\sigma dW_t + \mu_t\, dt) \\ &= \boldsymbol{\pi_t^\tau}\,\sigma d\widetilde{W}_t \\ X_0^\pi &= x_0 \end{aligned}$$

Shortfall Risk

Compare terminal wealth X_T with benchmark q

e.g. $q \sim x_0$ initial capital

Shortfall if $X_T < q$

Risk Measure $E_Q\left[(X_T - q)^-\right]$ where $Q \sim P$

Expected Loss

Special Cases

- $Q = \widetilde{P}$ **Present** Expected Loss (PEL)

  $\widetilde{E}\left[(X_T - q)^-\right]$ option price

  Basak, Shapiro (2001)

- $Q = P$ **Future** Expected Loss (FEL)

  $E\left[(X_T - q)^-\right]$ average additional capital

  Gabih, Grecksch, W. (2005)

Optimization Problem

Wealth equation $dX_t^\pi = \pi_t^\tau(\,\mu_t\,dt + \sigma\,dW_t),\quad X_0^\pi = x_0$

Strategy $\pi = (\pi_t)_{t\in[0,T]}$

Admissible strategies $\mathcal{A}(x_0) = \{(\pi_t) :$ **$\mathcal{F}^S$-adapted**

integrability conditions,

$X_t^\pi \geq 0,\ \forall t \in [0,T]\,\}$

Utility function $U : [0,\infty) \to \mathbb{R} \cup \{-\infty\}$

strictly increasing, concave (log-, power utility)

Optimization problem

$$V(x_0) = \sup_{\pi\in\mathcal{A}(x_0)} E\left[U(X_T^\pi)\right]$$

risk constraint $E_Q\left[(X_T^\pi - q)^-\right] \leq \varepsilon$

Decomposition of the OP: Full Information

Dynamic problem $$V(x_0) = \sup_{\pi \in \mathcal{A}(x_0)} E\left[U(X_T^{\pi})\right], \quad E_Q\left[(X_T^{\pi} - q)^-\right] \leq \varepsilon$$

Static problem $$V(x_0) = \sup_{\xi \in \mathcal{B}(x_0)} E\left[U(\xi)\right], \quad E_Q\left[(\xi - q)^-\right] \leq \varepsilon$$

where $\mathcal{B}(x_0) := \{\xi \geq 0 : \xi \text{ is } \mathcal{F}_T^S\text{-measurable}, \underbrace{\widetilde{E}[\xi] \leq x_0}_{\text{budget-constraint}} \}$

terminal wealth generated from initial capital in $(0, x_0]$

$\rightarrow$ **optimal terminal wealth** $\xi^* = f(Z_T)$

Representation problem

find a strategy $\pi \in \mathcal{A}(x_0)$ such that $\xi^* = X_T^{\pi}$

$\rightarrow$ **optimal strategy** π_t^*

The Case of Partial Information

Problem martingale density Z_t is $\mathcal{F}$- but **not $\mathcal{F}^S$-adapted**

Idea replace Z_t by its **filter** $\boldsymbol{\zeta_t} = E\left[Z_t | \mathcal{F}_t^S\right]$

Optimal terminal wealth $X_T^* = \xi^* = f(\boldsymbol{\zeta_T})$

Static Problem

$$V(x_0) = \sup_{\xi \in \mathcal{B}(x_0)} E\left[U(\xi)\right]$$

risk constraint $\quad \boldsymbol{E_Q\left[(\xi - q)^-\right] \leq \varepsilon}$

$\mathcal{B}(x_0) := \{\xi \geq 0 : \xi$ is $\mathcal{F}_T^S$-measurable, $\widetilde{E}\left[\xi\right] \leq x_0\}$

Choose the bound ε such that the risk constraint

- is binding $\quad \varepsilon \leq \varepsilon_{\max} = E_Q\left[(X_T^M - q)^-\right]$

  risk of the Merton portfolio

  (no risk constraint)

- can be fulfilled $\quad \varepsilon \geq \varepsilon_{\min} = \ldots \quad$ Gabih, Sass, W. (2006)

Optimal Terminal Wealth

Theorem ($Q = \widetilde{P}$ Present Expected Loss)

For $\varepsilon \in (\varepsilon_{\min}, \varepsilon_{\max})$ *the PEL-optimal terminal wealth is*

$$\xi^* = f(\zeta_T; y_1^*, y_2^*)$$

$$\text{where} \quad f(z; y_1, y_2) = \begin{cases} I(y_1 z) & \text{for} \quad z \in (0, z_l] \\ q & \text{for} \quad z \in (z_l, z_u] \\ I(\,(y_1 - y_2)z) & \text{for} \quad z \in (z_u, \infty). \end{cases}$$

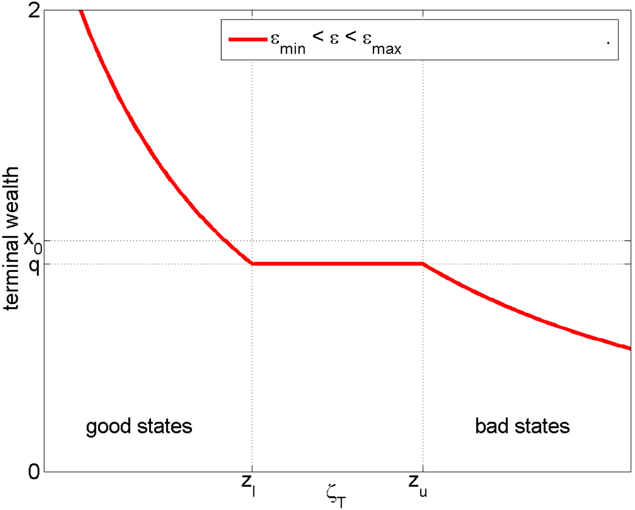
$$I = (U')^{-1}, \;\; z_l = \frac{U'(q)}{y_1} \;\; \text{and} \;\; z_u = \frac{U'(q)}{y_1 - y_2}.$$

The real numbers $y_1^*, y_2^* > 0$ *uniquely solve the equations*

$$\widetilde{E}\left[f(\zeta_T; y_1, y_2)\right] = x_0$$

$$E_Q\left[(f(\zeta_T; y_1, y_2) - q)^-\right] = \varepsilon.$$

Optimal Terminal Wealth: The Function $f(z; y_1, y_2)$



Computation of the Lagrange Multipliers y_1, y_2

System of nonlinear equations

$$\begin{aligned} \widetilde{E}\left[f(\zeta_T; y_1, y_2)\right] &= x_0 \\ E_Q\left[(f(\zeta_T; y_1, y_2) - q)^-\right] &= \varepsilon \end{aligned}$$

Existence & Uniqueness: Gabih, Sass, W. (2006)

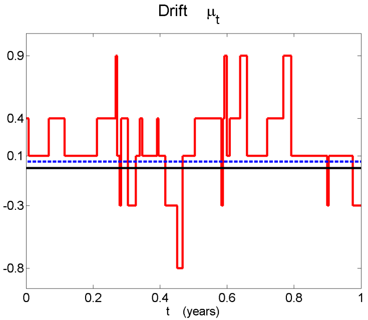
Solution requires

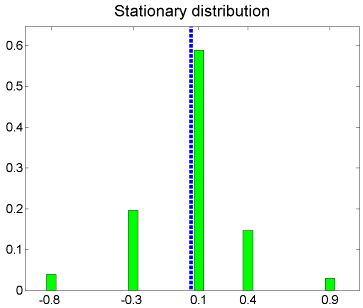
- Approximation of the expectation in on the left-hand sides using Monte-Carlo simulation
- Numerical methods for solving nonlinear equations

Example: Parameter of the Financial Market

$n = 1$ stock with volatility $\sigma = 0.25$

HMM for the drift μ_t with $d = 5$ states





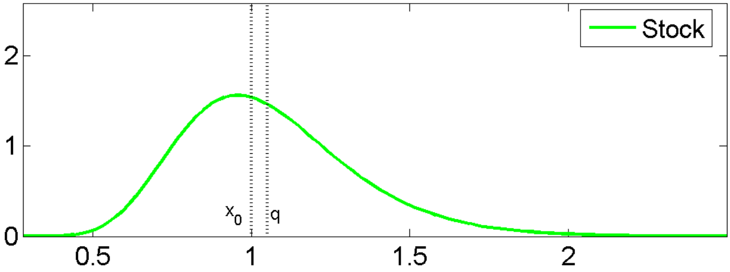
Ergodic mean $\overline{\mu} \approx$ **0.054**

Example: Parameter for the Portfolio Optimization

Horizon	$T = 1$ year
Utility function	$U(x) = 2x^{1/2} - 2$ (power utility)
Initial capital	$x_0 = 1$
Benchmark	$q = 1.05$ ($q > x_0$, portfolio insurance impossible)
Risk measure	Present Expected Loss $\widetilde{E}\left[(X_T - q)^-\right]$
Bound	$\varepsilon = 0.1$
Minimal Risk	$\varepsilon_{\min} = q - x_0 = 0.05$
Maximal Risk	$\varepsilon_{\max} \approx 0.248$ (i.e. $\varepsilon \approx 40\%$ of $\varepsilon_{\max}$)

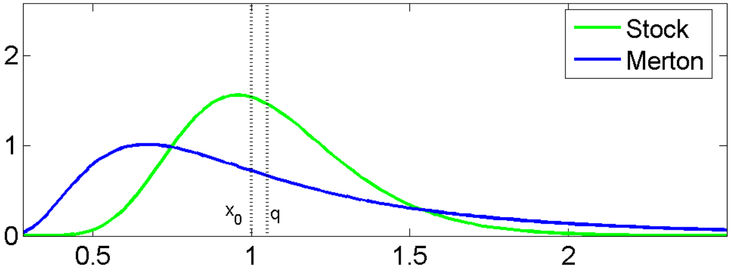
Monte-Carlo simulation: $N = 10^7$ realizations of $X_T^* = f(\zeta_T; y_1^*, y_2^*)$

Distribution of Terminal Wealth



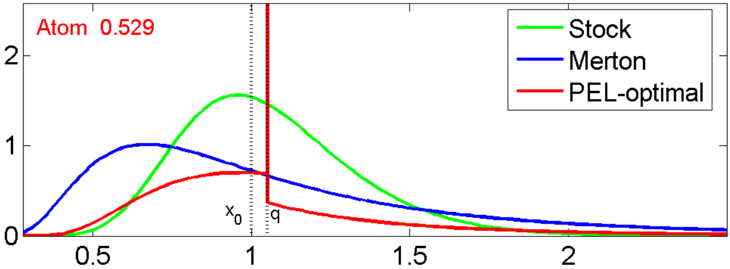
	Expected Utility	Risk
Stock	0.040	0.129

Distribution of Terminal Wealth



	Expected Utility	Risk
Stock	0.040	0.129
Merton	0.076	$0.248 = \varepsilon_{max}$

Distribution of Terminal Wealth



	Expected Utility	Risk
Stock	0.040	0.129
Merton	0.076	$0.248 = \varepsilon_{max}$
PEL-optimal	0.049	$0.100 = \varepsilon$

Optimal Strategy

Clark formula

Let $D_t\xi$ be the Malliavin derivative of the $\mathcal{F}_T^S$-measurable r.v. $\xi \in D_{1,1}$, then it holds

$$\xi = \widetilde{E}\,[\xi] + \int_0^T \widetilde{E}\left[(D_t\xi)^\tau | \mathcal{F}_t^S\right]\, d\widetilde{W}_t$$

For the Malliavin derivative of the optimal terminal wealth

$\xi = f(\zeta_T) = f(\zeta_T; y_1^*, y_2^*)$ it holds $D_t\, f(\zeta_T) = f'(\zeta_T)\, D_t\zeta_T$

Wealth equation $\xi = x_0 + \int_0^T (\pi_t^*)^\tau \sigma\, d\widetilde{W}_t$

Comparison of coefficients $\Rightarrow$ optimal strategy is

$$\pi_t^* = \sigma^{-\tau}\, \widetilde{E}\left[f'(\zeta_T)\, D_t\zeta_T \,|\, \mathcal{F}_t^S\right]$$

Computation of the Optimal Strategy

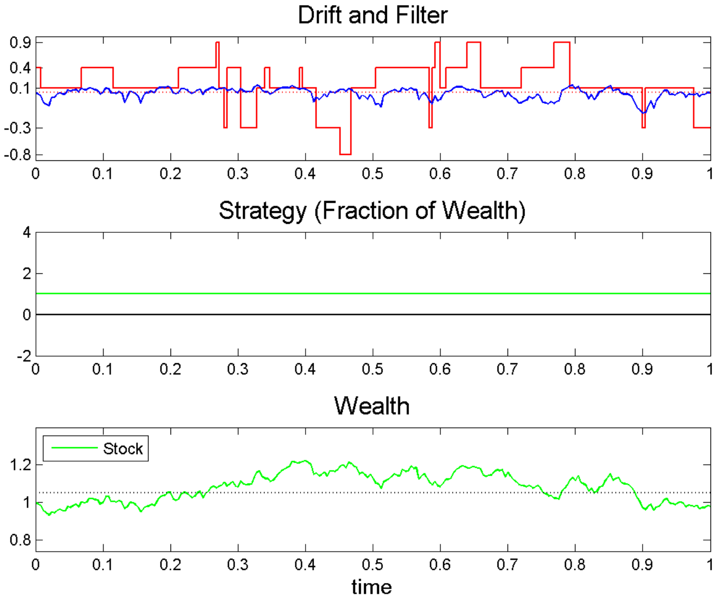
$$\pi_t^* \quad = \quad \sigma^{-\tau}\, \widetilde{E}\left[f'(\zeta_T)\, D_t\zeta_T \,|\, \mathcal{F}_t^S\right] \qquad (*)$$

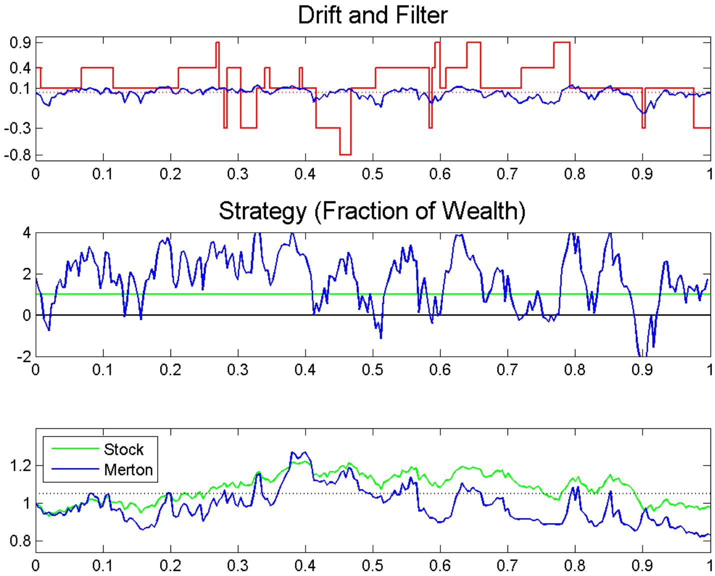
requires

- Numerical solution of SDE's for the Malliavin Derivative $D_t\zeta_T$
- Approximation of the conditional expectation in $(*)$ using Monte-Carlo simulation

  Generate $N = 10^3$ realizations of ζ_T and $D_t\zeta_T$

$$\vdots$$


Drift and Filter
0.9
0.4
0.1
-0.3
-0.8
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Strategy (Fraction of Wealth)
4
2
0
-2
Wealth
Stock
1.2
1
0.8
time


Drift and Filter
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0.7
0.8
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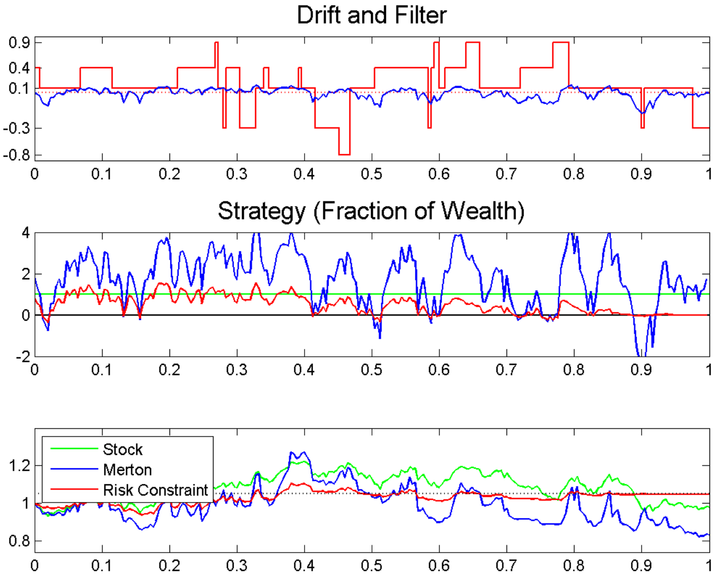
Strategy (Fraction of Wealth)
4
2
0
-2
Stock
Merton
1.2
1
0.8


Drift and Filter
0.9
0.4
0.1
-0.3
-0.8
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Strategy (Fraction of Wealth)
4
2
0
-2
Stock
Merton
Risk Constraint
1.2
1
0.8

Trading in Discrete Time

(A) Actual wealth

$$X_t^* = x_0 + \int_0^t (\pi_s^*)^\tau \, dR_s \approx x_0 + \sum_{0 \le s_i < t} (\widehat{\pi}_{s_i})^\tau \, \Delta R_{s_i} =: X_t^A$$

where $\widehat{\pi}_{s_i}$ is the Monte-Carlo approximation of $\pi_{s_i}^*$

(B) Theoretical optimal wealth

$$X_t^* = \widetilde{E}\left[X_T^* | \mathcal{F}_t^S\right] = \widetilde{E}\left[f(\zeta_T; y_1^*, y_2^*) | \mathcal{E}_t\right]$$

Given the unnormalized filter at time t is $\mathcal{E}_t = x$ we find

$$X_t^* = \widetilde{E}\left[f(\zeta_T^{t,x}; y_1^*, y_2^*)\right] \approx X_t^B \quad \text{(by Monte-Carlo approximation)}$$

Can be evaluated without computing the optimal strategy

If $X_t^A \neq X_t^*$ then trading according to π^* is no longer optimal

Updating of the Optimal Strategy

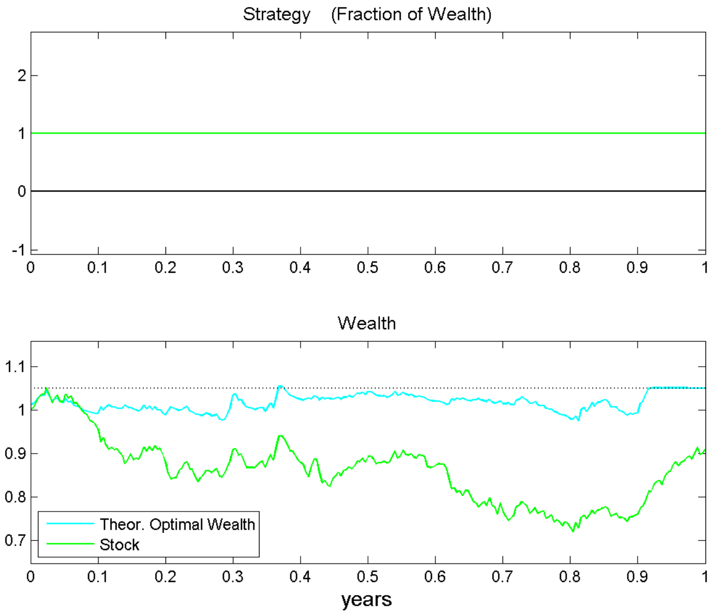
If we observe a "critical deviation" $|X_t^A - X_t^B| > \delta$

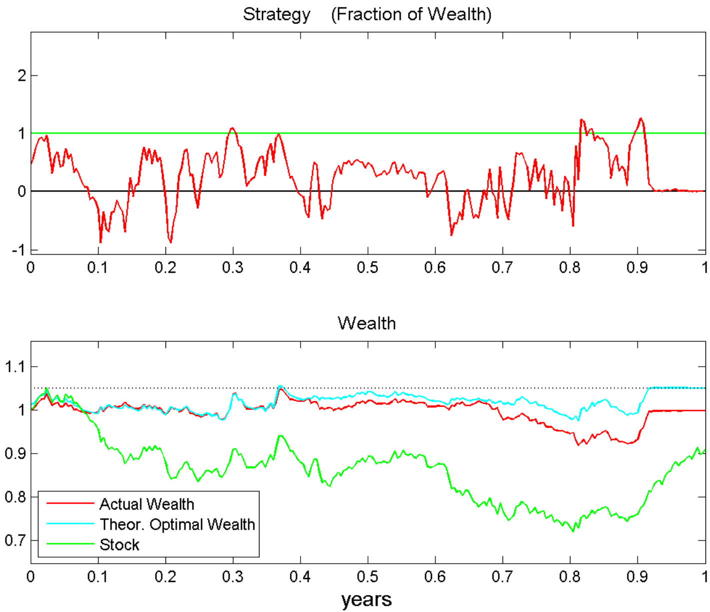
we set up a new optimization problem with

Horizon	$T - t$	
Initial capital	X_t^A	(actual wealth)
Risk bound	Expected Loss of X_T^* at time t given $\mathcal{F}_t^S$	
	$\varepsilon_t^* = E_Q\left[(X_T^* - q)^- \| \mathcal{F}_t^S\right]$	(= option price for PEL)

Compute new Lagrange multipliers y_1^t, y_2^t by solving

$$\begin{aligned} \widetilde{E}\left[f(\zeta_T; y_1, y_2) | \mathcal{F}_t^S\right] &= X_t^A \\ E_Q\left[(f(\zeta_T; y_1, y_2) - q)^- | \mathcal{F}_t^S\right] &= \varepsilon_t^*. \end{aligned}$$


Strategy (Fraction of Wealth)
Wealth
Theor. Optimal Wealth
Stock
years


Strategy (Fraction of Wealth)
2
1
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-1
0
0.1
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0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Wealth
1.1
1
0.9
0.8
0.7
Actual Wealth
Theor. Optimal Wealth
Stock
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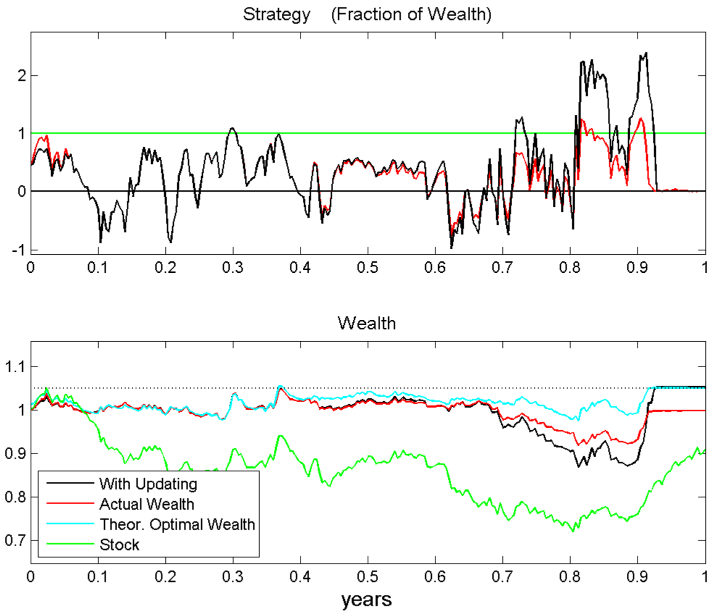
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
years


Strategy (Fraction of Wealth)
2
1
0
-1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Wealth
1.1
1
0.9
0.8
0.7
With Updating
Actual Wealth
Theor. Optimal Wealth
Stock
years

Conclusion

- Dynamic portfolio optimization under risk constraint

$$V(x_0) = \sup_{\pi \in \mathcal{A}(x_0)} E\left[U(X_T^\pi)\right], \qquad E_Q\left[(X_T^\pi - q)^-\right] \le \varepsilon$$

- Partial information on the drift (Hidden Markov Model)

  X_T^* as a function of ζ_T, the filter for the martingale density Z_T

  π_t^* depends on Malliavin derivative $D_t X_T^*$

- Optimal strategies can be computed using Monte-Carlo simulations

- For references see www.fh-zwickau.de/~raw