Stopping of Integral Functionals of Diffusions and a “No-Loss” Free Boundary Formulation

M. Urusov

Quantitative Products Laboratory
Technical University Berlin

Based on joint works with D. Belomestny and L. Rüschendorf

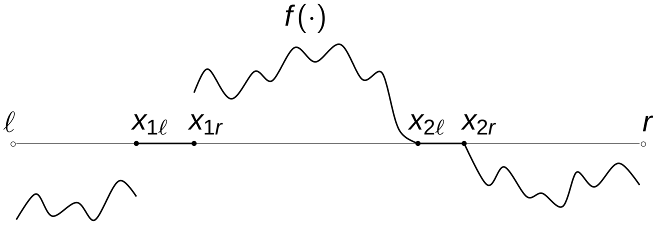
AMaMeF Conference, Vienna, September 17–22, 2007

Formulation of the Problem

$$V^*(x) = \sup_\tau \mathsf{E}_x \int_0^\tau e^{-\Lambda_u} f(X_u)\, du \qquad \text{(OS)}$$

- $dX_t = \mu(X_t)\, dt + \sigma(X_t)\, dW_t, \quad \mathsf{P}_x(X_0 = x) = 1$
- $X_t \in J := (\ell, r)$
- $\Lambda_t = \int_0^t \lambda(X_u)\, du$
- $\lambda \colon J \to [0, \infty)$

Terminology



In this case we say that f has a two-sided form

Examples:
Graversen, Peskir, and Shiryaev (2000)
Karatzas and Ocone (2002)

Assumptions on μ, σ, f, and λ

Assumptions on μ and σ:

$$\sigma(x) \neq 0 \;\; \forall x \in J, \quad \frac{1}{\sigma^2} \in L^1_{loc}(J), \quad \frac{\mu}{\sigma^2} \in L^1_{loc}(J)$$

Assumptions on f and λ:

$$\frac{f}{\sigma^2}, \; \frac{\lambda}{\sigma^2} \in L^1_{loc}(J) \qquad (*)$$

Explanation of $(*)$:

$(*) \iff (F_t)$ and (Λ_t) are well defined and finite until ζ

$F_t = \int_0^t f(X_u)\,du, \quad \Lambda_t = \int_0^t \lambda(X_u)\,du$

ζ is the explosion time of X

Outline

Notation

For $\alpha, \beta \in J$, $\alpha < \beta$, we set

$$\tau_{\alpha,\beta} = \inf\{t \geq 0 \colon X_t \leq \alpha \text{ or } X_t \geq \beta\}$$

Standard free boundary

(SFB):

$$\frac{\sigma^2(x)}{2}V''(x) + \mu(x)V'(x) - \lambda(x)V(x) = -f(x), \quad x \in (\alpha, \beta)$$

$$V(x) = 0, \quad x \in J \setminus (\alpha, \beta)$$

$$V'_+(\alpha) = V'_-(\beta) = 0$$

The aim:

If (V, α, β) is a solution, then $V = V^*$ and $\tau_{\alpha,\beta}$ is optimal in (OS)

Modifying free boundary

$$\frac{\sigma^2(x)}{2}V''(x) + \mu(x)V'(x) - \lambda(x)V(x) = -f(x)$$

for ν_L-a.a. $x \in (\alpha, \beta)$

$$V(x) = 0, \quad x \in J \setminus (\alpha, \beta)$$

$$V'_+(\alpha) = V'_-(\beta) = 0$$

The aim:

If (V, α, β) is a solution, then $V = V^*$ and $\tau_{\alpha,\beta}$ is optimal in (OS)

"No-loss" free boundary

V' is absolutely continuous on $[\alpha, \beta]$

$$\frac{\sigma^2(x)}{2} V''(x) + \mu(x) V'(x) - \lambda(x) V(x) = -f(x)$$

$$\text{for } \nu_L\text{-a.a. } x \in (\alpha, \beta)$$

$$V(x) = 0, \quad x \in J \setminus (\alpha, \beta)$$

$$V'_+(\alpha) = V'_-(\beta) = 0$$

The aim:

If (V, α, β) is a solution, then $V = V^*$ and $\tau_{\alpha,\beta}$ is optimal in (OS)

“No-loss” free boundary

(FB):

$$V' \text{ is absolutely continuous on } [\alpha, \beta]$$

$$\frac{\sigma^2(x)}{2} V''(x) + \mu(x)V'(x) - \lambda(x)V(x) = -f(x)$$

$$\text{for } \nu_L\text{-a.a. } x \in (\alpha, \beta)$$

$$V(x) = 0, \quad x \in J \setminus (\alpha, \beta)$$

$$V'_+(\alpha) = V'_-(\beta) = 0$$

The aim:

If (V, α, β) is a solution, then $V = V^*$ and $\tau_{\alpha,\beta}$ is optimal in (OS)

Main results

Theorem (Verification Theorem)

Suppose f has a two-sided form.
Let (V, α, β) *be a solution of (FB). Then*

- *it is unique*
- $V^* = V$
- $\tau_{\alpha,\beta}$ *is a unique optimal stopping time in (OS)*

Theorem ((FB) is "no-loss")

If (OS) has an optimal stopping time of the form τ_{α^*,β^*}*,*
then (V^*, α^*, β^*) *is a solution of (FB)*

Related paper: Lamberton and Zervos (2006)

The assumption that f has a two-sided form is essential for the verification theorem

Example

- f does not have a two-sided form
- (V, α, β) is a solution of (FB)
- $V \neq V^*$
- $\tau_{\alpha,\beta}$ is not optimal in (OS)

The subclass of stopping problems

In the sequel $J = \mathbb{R}$, $\mu \equiv 0$, $\lambda \equiv 0$

Characteristic conditions

(A_1): $h(\infty) > h(-\infty)$

(A_2): If $h(\infty) < h(x_{1\ell})$, then $\int_{a_{h(\infty)}}^{\infty} H(y, h(\infty))\, dy < 0$

(A_3): If $h(-\infty) > h(x_{2r})$, then $\int_{-\infty}^{b_{h(-\infty)}} H(y, h(-\infty))\, dy > 0$

Here

$$h(x) := -\int_0^x \frac{2f(y)}{\sigma^2(y)}\, dy \quad \text{and} \quad H(x, c) := h(x) - c$$

Classification

Case 0: (A_1)–(A_3) hold

Case 1: (A_1) does not hold

Case 2: (A_1) holds and (A_2) does not hold

Case 3: (A_1) holds and (A_3) does not hold

Case 0

- (FB) has a unique solution
- (OS) has a unique optimal stopping time and it is two-sided
- The value function and the optimal stopping time are found explicitly

Case 1

- (FB) has no solution
- (OS) has no optimal stopping time
- The value function is found explicitly
- It can be either finite or identically infinite

Case 2

- (FB) has no solution
- (OS) has a unique optimal stopping time, it is one-sided with the unbounded from below stopping region
- The value function is always finite and is found explicitly, together with the optimal stopping time

Case 3

is symmetric to case 2

Thank you for your attention!

Graversen, S. E., G. Peskir, and A. N. Shiryaev (2000).
Stopping Brownian motion without anticipation as close as possible to its ultimate maximum.
Theory Probab. Appl. 45(1), 125–136.

Karatzas, I. and D. Ocone (2002).
A leavable bounded-velocity stochastic control problem.
Stochastic Process. Appl. 99(1), 31–51.

Lamberton, D. and M. Zervos (2006).
On the problem of optimally stopping a one-dimensional Itô diffusion.
Submitted.