Contagious default: application of methods of Statistical Mechanics in Finance

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Outline

- *The financial problem :* credit risk and the modeling of contagion

- *The interacting particle system model*

- *The main results for the particle system*

  i) Asymptotics when the number of particles $N \to \infty$
  ii) Equilibria of the limiting dynamics
  iii) Finite volume approximations

- *Back to Finance :* large portfolio losses in a credit risky environment with contagion and default clustering.

The financial problem : Credit risk

- Risk faced by a financial institution holding a portfolio of positions issued by a (large) number of firms that may default.

  i) *Default* may be *contagious*
  ii) There may be a *clustering of defaults* (many defaults happen in a short time)

  Losses may therefore be large and we want to address this problem in the above context *(contagion and clustering)*.

- *Reduced-form or intensity-based approach*

Contagion *(interacting intensities)*

- *To describe propagation of financial distress in a network of firms linked (directly or indirectly) by business relationships one possibility is via interacting intensities.*

  $\longrightarrow$ A natural way to obtain interacting intensities is to let the default intensities depend on a *common exogenous macroeconomic factor process* X_t, i.e. for the generic $j-$th firm one postulates

$$\lambda_t^j = \lambda^j(X_t)$$

- Given $\lambda_t^j = \lambda^j(X_t)$

  i) If X_t is observable and has jumps in common with the point process counting the defaults $\longrightarrow$ *direct contagion (counterparty risk)*

  ii) If X_t is unobservable, but its distribution is successively updated on the basis of the observed default history $\longrightarrow$ *information induced contagion*.

$\longrightarrow$ *Interacting intensity models are currently those mostly investigated and they are motivated by the empirical observation that default intensities are correlated with macroeconomic factors.*

- However (quoting from Jarrow and Yu (2001)):

  "A default intensity that depends linearly on a set of smoothly varying (exogenous) macroeconomic variables is unlikely to account for the clustering of defaults around an economic recession".

- Furthermore, one might also want to describe the general health of a network of firms by *endogenous financial indicators* thereby viewing *a credit crisis as a microeconomic phenomenon* and so possibly also arrive at explaining *default clustering*.

  → *Interacting particle system models from Statistical mechanics may allow to adequately address the above issues.*

The interacting particle system model

- *A mean-field interacting model of the Curie-Weiss type*; a simple model to describe dynamically the credit quality of firms.

- *The "credit state" of each firm is identified by two variables* (σ, ω) $((\sigma_i, \omega_i)$: *state of* $i-$*th firm* $i = 1, \cdots, N$*)*.

  $\rightarrow$ σ : a *"rating class/financial distress indicator" (a low value reflects a bad rating class, i.e. a higher probability of not being able to pay back obligations)*.

  $\rightarrow$ ω : a more fundamental indicator of the *financial health of the firm*; (it represents a local random environment and is typically not directly observable from the market).

- At a first level assume $(\sigma_i, \omega_i) \in \{-1, +1\}^2$ *(generalization to a generic finite number of possible values rather straightforward)*

- No explicit "default state" (could be $\sigma_i = -1$).
  Always need a positive probability that the firm can exit from the state where σ_i takes its lowest possible value.

- For the time evolution on a generic interval $[0, T]$ of the "state" of the particle system, i.e. $(\sigma_i(t), \omega_i(t))_{i=1,\cdots,N} \in \mathcal{D}^{2N}[0, T]$ we need to specify the stochastic dynamics for the transitions $\sigma_i \rightarrow -\sigma_i, \ \omega_i \rightarrow -\omega_i$.

- *The mean-field assumption leads to letting the interaction depend on the global health indicator (endogenous global factor)*

$$m_N^\sigma(t) := \frac{1}{N} \sum_{i=1}^{N} \sigma_i(t)$$

- The vehicle of interaction/contagion is given by

$$\underset{fundam.\ indic.}{\omega_i} \quad \longrightarrow \quad \underset{rating\ class}{\sigma_i} \quad \longrightarrow \quad \underset{global\ health\ indic.}{m_N^\sigma} \quad \longrightarrow \omega_j$$

Transition intensities for the particle system

$$\begin{cases} \sigma_i \to -\sigma_i \quad \textit{with intensity} \;\; \lambda_i \;:= e^{-\beta\sigma_i\omega_i}, \quad \beta > 0 \\ \omega_j \to -\omega_j \quad \textit{with intensity} \;\; \mu_j \;:= e^{-\gamma\omega_j m_N^\sigma}, \quad \gamma > 0 \end{cases}$$

β, γ are parameters indicating the strength of the interaction (*This induces a "symmetry" in the model*).

$\rightarrow$ The resulting transition intensity matrix can be taken as *infinitesimal generator* L of a *continuous-time Markov chain* with state space $\{-1,+1\}^{2N}$ that acts on $f : \{-1,1\}^{2N} \to \mathbb{R}$ as

$$Lf(\sigma,\omega) = \sum_{i=1}^{N} \lambda_i \nabla_i^\sigma f(\sigma,\omega) + \sum_{j=1}^{N} \mu_j \nabla_j^\omega f(\sigma,\omega)$$

where $\nabla_i^\sigma f(\sigma,\omega) = f(\sigma^i,\omega) - f(\sigma,\omega)$; $\quad \nabla_j^\omega f(\sigma,\omega) = f(\sigma,\omega^j) - f(\sigma,\omega)$ and $\sigma^i = (\sigma_1, ..., \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, ..., \sigma_N)$; analogously for ω^j.

- Unlike many mean field models in Statistical mechanics our *model is non-reversible*.

$\longrightarrow$ *An explicit formula for the stationary (in time) distribution is not available.*

$\longrightarrow$ We shall rather

A. Look for the *limit* $(N \to \infty)$ *dynamics* of the system on the path space *(via a LLN based on a Large Deviations Principle)*;

B. Study the *equilibria of the limiting dynamics*;

C. Describe *"finite volume approximations"* *(for large but finite N) via a Central Limit type result.*

$\longrightarrow$ *Non-standard versions of LLN and CLT*

A. Limit for $N \to \infty$ (Law of Large Numbers)

- Let ($\delta_{\{\cdot\}}$ denotes the Dirac measure)

$$\rho_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\{\sigma_i[0,T],\omega_i[0,T]\}}$$

be the sequence of empirical (random) measures on the space $\mathcal{M}_1(\mathcal{D}^2[0,T])$ endowed with the weak convergence topology.

- For a probability measure $q \in \mathcal{M}_1(\{-1,1\}^2)$ let

$$m_q^\sigma := \sum_{\sigma,\omega=\underline{+}1} \sigma\, q(\sigma,\omega)$$

(expected health under q).

Theorem 1. Let $(\sigma(t), \omega(t))$ be the Markov process corresponding to the generator $Lf(\sigma, \omega)$ and with initial distribution s.t. $(\sigma_i(0), \omega_i(0))$, $i = 1, \cdots, N$ are i.i.d. with law ℓ.

i) There exists $Q^* \in \mathcal{M}_1(\mathcal{D}^2[0,T])$ s.t. $\rho_N \to Q^*$ a.s. in the weak topology;

ii) if $q_t \in \mathcal{M}_1(\{-1,1\}^2)$ is the marginal distribution of Q^* at time t, then it is the unique solution of the *McKean-Vlasov equation (MKV)*

$$\begin{cases} \frac{\partial q_t}{\partial t} & = & \mathcal{L} q_t \,, \quad t \in [0,T] \\ q_0 & = & \ell \end{cases}$$

with $\mathcal{L} q(\sigma, \omega) = \nabla^{\sigma} \left[e^{-\beta \sigma \omega} q(\sigma, \omega) \right] + \nabla^{\omega} \left[e^{-\gamma \omega m_q^{\sigma}} q(\sigma, \omega) \right]$

B. Large time behavior of the limiting $(N \to \infty)$ dynamics

- A measure μ on $\{-1,1\}^2$ is completely specified by

$m_\mu^\sigma := \sum_{\sigma,\omega=\pm 1} \sigma\, \mu(\sigma,\omega), m_\mu^\omega := \sum_{\sigma,\omega=\pm 1} \omega\, \mu(\sigma,\omega), m_\mu^{\sigma\omega} := \sum_{\sigma,\omega=\pm 1} \sigma\omega\, \mu(\sigma,\omega)$

Write $m_t^\sigma = m_{q_t}^\sigma$ (analogously for m_t^ω, $m_t^{\sigma\omega}$)

- *(MKV) can be reduced to determining a solution of*

$$(\dot m_t^\sigma, \dot m_t^\omega) = V(m_t^\sigma, m_t^\omega) \qquad \text{(mkv)} \quad \text{with}$$

$$V(x,y) := (2\sinh(\beta)y - 2\cosh(\beta)x, 2\sinh(\gamma x) - 2y\cosh(\gamma x))$$

$\rightarrow$ *To analyze in (MKV) equilibria and their stability it suffices to analyze (mkv)*

Theorem 2.

i) Suppose $\gamma \le \frac{1}{\tanh(\beta)}$. Then equation (mkv) has $(0,0)$ as a unique equilibrium solution, which is *globally asymptotically stable*, i.e. for every initial condition (m_0^σ, m_0^ω), we have

$$\lim_{t \to +\infty} (m_t^\sigma, m_t^\omega) = (0,0).$$

ii) For $\gamma < \frac{1}{\tanh(\beta)}$ the equilibrium $(0,0)$ is *linearly stable*, i.e. $DV(0,0)$ *(the Jacobian matrix)* has strictly negative eigenvalues. For $\gamma = \frac{1}{\tanh(\beta)}$ the linearized system has a neutral direction, i.e. $DV(0,0)$ has one zero eigenvalue.

iii) For $\gamma > \frac{1}{\tanh(\beta)}$ the point $(0,0)$ is still an equilibrium for (mkv), but it is a saddle point for the linearized system, i.e. the matrix $DV(0,0)$ has two nonzero real eigenvalues of opposite sign. Moreover (mkv) has *two linearly stable solutions* (m_*^σ, m_*^ω), $(-m_*^\sigma, -m_*^\omega)$, where m_*^σ is the unique strictly positive solution of the equation

$$x = \tanh(\beta)\tanh(\gamma x),$$

and

$$m_*^\omega = \frac{1}{\tanh(\beta)} m_*^\sigma$$

iv) For $\gamma > \frac{1}{\tanh(\beta)}$, the *phase space* $[-1,1]^2$ *is bi-partitioned* by a smooth curve Γ containing $(0,0)$ such that $[-1,1]^2 \setminus \Gamma$ is the union of two disjoint sets Γ^+, Γ^- that are open in the induced topology of $[-1,1]^2$. Moreover

$$\lim_{t \to +\infty} (m_t^\sigma, m_t^\omega) = \begin{cases} (m_*^\sigma, m_*^\omega) & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma^+ \\ (-m_*^\sigma, -m_*^\omega) & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma^- \\ (0,0) & \text{if } (m_0^\sigma, m_0^\omega) \in \Gamma. \end{cases}$$

- The fact that the limiting $(N \to \infty)$ dynamics may have multiple stable equilibria implies that our system exhibits what is called *phase transition:*.

  $\to$ *The asymptotic ($N \to \infty$) behavior of the system changes depending on the values of the parameters (and of the initial conditions).*

  $\to$ The *effects of phase transition* for the system with finite N can be seen on *different time scales* in different ways.

- On regular time-scales (of order O(1)) the following occurs: *for certain values of the initial condition the system is driven towards the asymptotic symmetric equilibrium state* $(0, 0)$ *where half of the firms are in good financial health.*

  After a certain time (depending on the initial condition) the system is captured by an unstable direction of this neutral equilibrium and moves towards a stable asymmetric equilibrium. *During this transition the volatility of the system (will be defined below) increases sharply* before decaying to a stationary value.

$\rightarrow$ *This phenomenon can be interpreted as a credit crisis and may account for default clustering.*

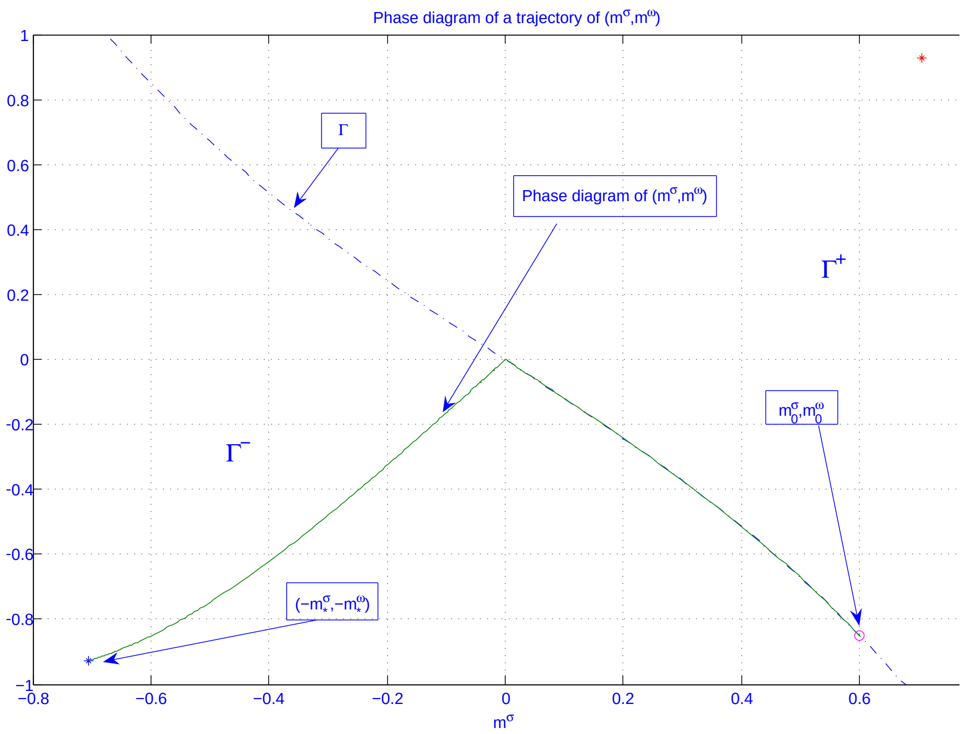
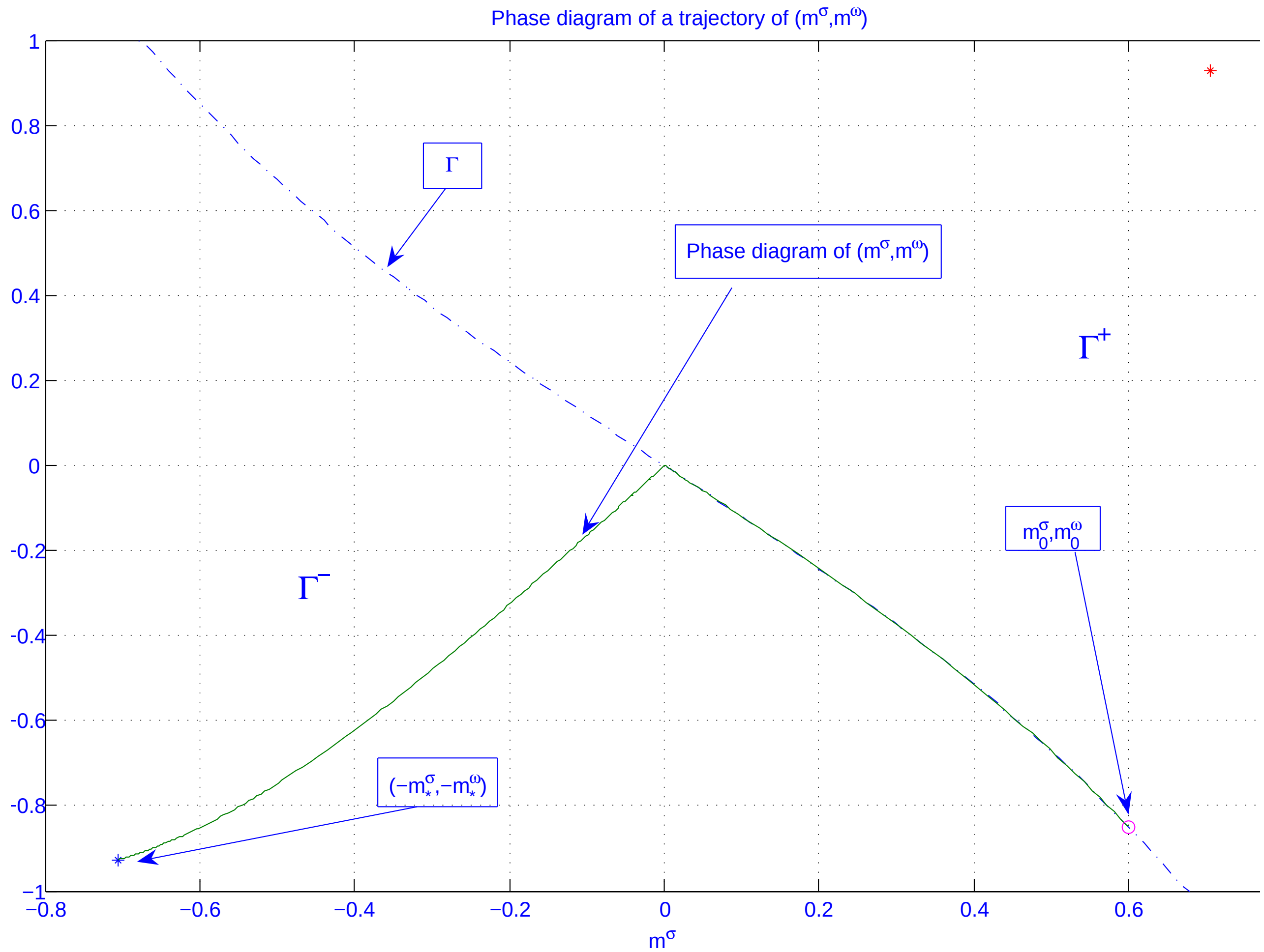


Figure 1: $\beta=1$, $\gamma=2.3$, $\gamma_c=1/tanh(\beta)\approx 1.313$



Phase diagram of a trajectory of (m^σ,m^ω)
Γ
Phase diagram of (m^σ,m^ω)
Γ+
Γ−
m0σ,m0ω
(−m*σ,−m*ω)
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
mσ

Figure 2: β=1.5, γ=2.1, γ_c=1/$tanh(\beta)$≈1.105

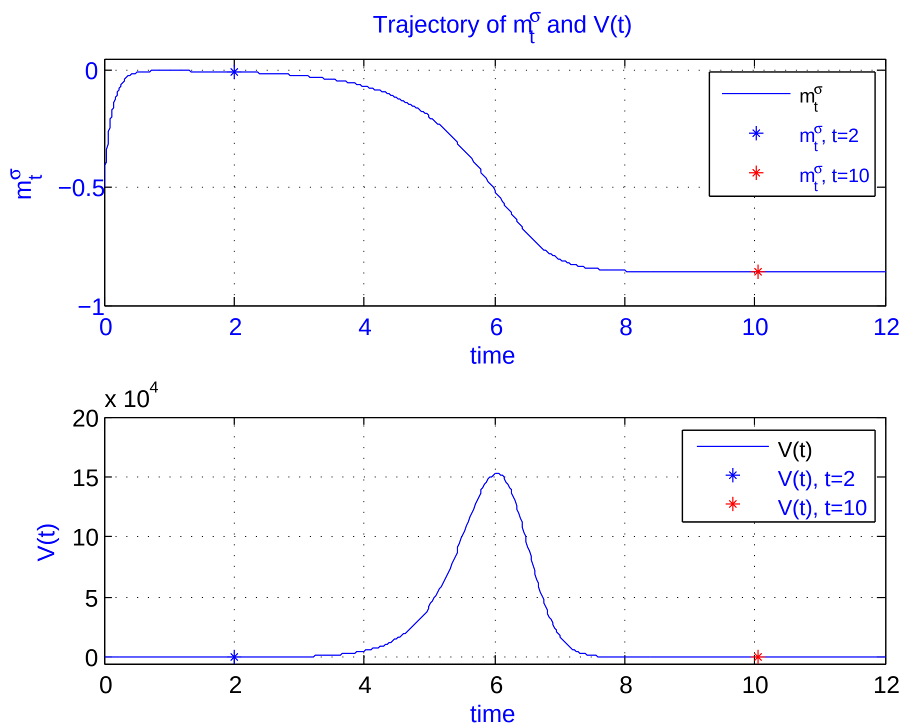
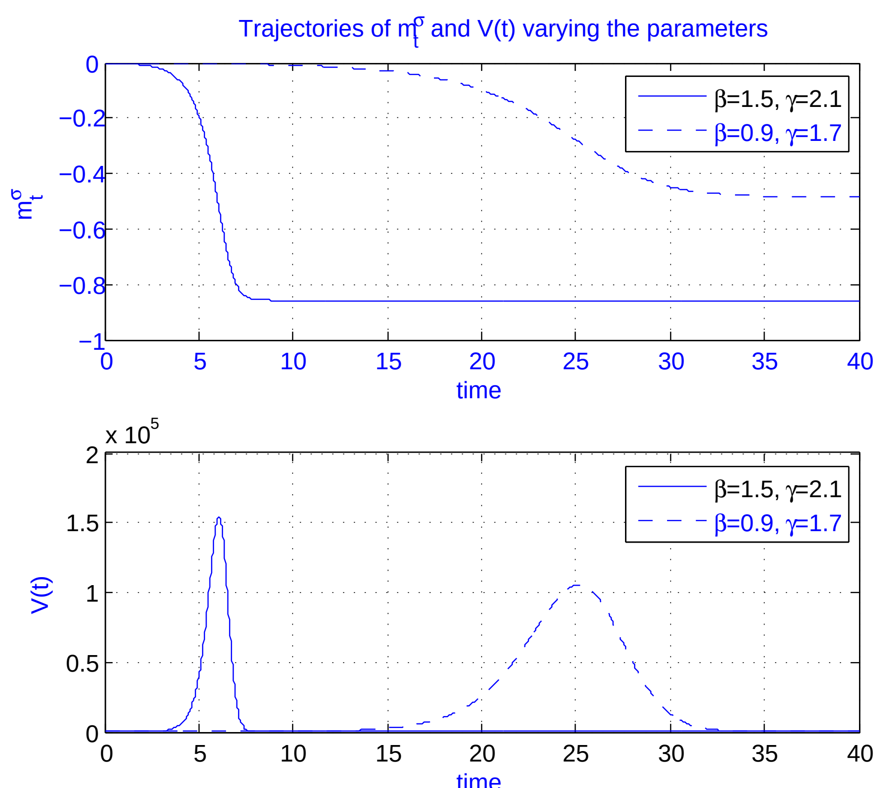


Figure 3: $\gamma_c \approx 1.105$; $\gamma_c \approx 1.396$



C. Analysis of the fluctuations

- Concerns the asymptotic distribution of $(\rho_N - Q^*)$.

  $\rightarrow$ *Recall that $\rho_N(t)$, being a measure on $\{-1,1\}^2$, is characterized by*

$$m^{\sigma}_{\rho_N}(t),\ \ m^{\omega}_{\rho_N}(t),\ \ m^{\sigma\omega}_{\rho_N}(t)$$

- With $A(t), D(t)$ appropriate matrices depending on β, γ and $m^{\sigma}_t, m^{\omega}_t, m^{\sigma\omega}_t$ one has the following

Theorem 3. Let

$$\begin{cases} x_N(t) & = & \sqrt{N}\left(m^{\sigma}_{\rho_N}(t) - m^{\sigma}_t\right) \\ y_N(t) & = & \sqrt{N}\left(m^{\omega}_{\rho_N}(t) - m^{\omega}_t\right) \\ z_N(t) & = & \sqrt{N}\left(m^{\sigma\omega}_{\rho_N}(t) - m^{\sigma\omega}_t\right) \end{cases}$$

Then $(x_N(t), y_N(t), z_N(t)) \overset{N\to\infty}{\longrightarrow} (x(t), y(t), z(t))$ in the sense of weak convergence of stochastic processes, where $(x(t), y(t), z(t))$ is a centered Gaussian process, unique solution of the linear SDE

$$\begin{pmatrix} dx(t) \\ dy(t) \\ dz(t) \end{pmatrix} = A^*(t) \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} dt + D(t) \begin{pmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{pmatrix}$$

where B_1, B_2, B_3 are independent Brownian motions and $(x(0), y(0), z(0))$ is a centered Gaussian.

→ *The asymptotic, for* $N \to \infty$*, distribution of* $(x_N(t), y_N(t), z_N(t))$ *is thus, for each fixed* t*, a centered Gaussian with covariance matrix* Σ_t *- the volatility referred to earlier - satisfying (asymptotics in* t *depend upon* γ*)*

$$\frac{d\Sigma_t}{dt} = A(t)\,\Sigma_t + \Sigma_t\,A^*(t) + DD^*(t)$$

Corollary 1: $\sqrt{N}\ \left[m^\sigma_{\rho_N}(t) - m^\sigma_t\right] \ \xrightarrow{D}\ \mathcal{N}(0, \Sigma^x_t)$ so that (notice that $m^\sigma_N(t) = m^\sigma_{\rho_N}(t)$)

$$P(m^\sigma_N(t) \geq \alpha) \approx \Phi\left(\frac{\sqrt{N}m^\sigma_t - \sqrt{N}\alpha}{\sqrt{\Sigma^x_t}}\right)$$

$(\Phi(\cdot)$ cumulative standard Gaussian).

Portfolio losses

- A bank holds a portfolio of financial positions issued by the N firms.

- *Random loss* for the $i-th$ position at time t:

$$L_i(t) \in \mathbb{R}^+; \quad i = 1, ..., N$$

- *Aggregated losses* are $L^N(t) = \sum_{i=1}^N L_i(t)$

- More specifically, let

$$G_x(u) := P\{L_i(t) \le u \mid \sigma_i(t) = x\}\,, \quad x \in \{-1, +1\}$$

(*homogeneity with respect to i and t*) and

$$\ell_1 := E\{L_i(t) \mid \sigma_i(t) = 1\} < E\{L_i(t) \mid \sigma_i(t) = -1\} := \ell_{-1}$$

$\rightarrow$ *one expects to loose more when in financial distress.*

Furthermore,

$$v_1 := Var\{L_i(t) \mid \sigma_i(t) = 1\}; v_{-1} := Var\{L_i(t) \mid \sigma_i(t) = -1\}$$

Example 1

- Portfolio consisting of N positions of 1 unit due at time T *(defaultable bonds)*.

$$L_i(T) = L(\sigma_i(T)) = \begin{cases} 1 & if \quad \sigma_i(T) = -1 \\ 0 & if \quad \sigma_i(T) = 1 \end{cases}$$

$$\longrightarrow \qquad L^N(T) = \sum_{i=1}^{N} \frac{1-\sigma_i(T)}{2} \; = \; \frac{N(1-m^{\sigma}_N(T))}{2}$$

$$\longrightarrow \qquad P\{L^N(T) \geq \alpha\} = P\left\{m^{\sigma}_N(T) \leq 1 - \frac{2\alpha}{N}\right\}$$

apply Corollary 1

A further result

- Let

$$L(t) := \frac{(\ell_1 - \ell_{-1})}{2} m_t^\sigma + \frac{(\ell_1 + \ell_{-1})}{2}$$

$$V(t) := \frac{(\ell_1 - \ell_{-1})^2 \Sigma_t^x}{4} + \frac{(1 + m_t^\sigma)\, v_1}{2} + \frac{(1 - m_t^\sigma)\, v_{-1}}{2}$$

Theorem 4: When the distribution of $L_i(t)$ depends on $\sigma_i(t)$,

$$\sqrt{N}\left(\frac{L^N(t)}{N} - L(t)\right) \xrightarrow{D} \mathcal{N}\left(0, V(t)\right)$$

Corollary 2: In the setting of Theorem 4 it follows

$$P\left\{L^N(T) \geq \alpha\right\} \sim \Phi\left(\frac{NL(T) - \alpha}{\sqrt{N}\sqrt{V(T)}}\right)$$

Example 2 (Bernoulli mixture model)

- As before but with

$$L_i(T) = L(\sigma_i(T);\Psi) = \begin{cases} 1 & \textit{with prob} \quad P(\sigma_i(T);\Psi) \\ \\ 0 & \textit{with prob} \quad 1 - P(\sigma_i(T);\Psi) \end{cases}$$

where Ψ is an exogenous random factor.

$\rightarrow$ $\ell_1 = P(1;\Psi), v_1 = P(1;\Psi)(1 - P(1;\Psi))$ (analogously for ℓ_{-1}, v_{-1})

- *A possible specification is*

$$P(\sigma;\Psi) = 1 - \exp\{-k_1\Psi - k_2(1-\sigma)/2 - k_3\}$$

with $k_i \geq 0$ *and* $\Psi \sim \Gamma(\alpha;\kappa)$.*(The prob. for* $L_i(T) = 1$ *is bigger for* $\sigma_i(T) = -1$ *than for* $\sigma_i(T) = 1$*).*

- *Here* $\ell_1, \ell_{-1}, v_1, v_{-1}$ *and thus also* $L(t)$ *and* $V(t)$ *depend on the value* ψ *taken by the Gamma-type r.v.* Ψ. *Denote the latter by* $L(t;\psi)$, $V(t,\psi)$.

$\longrightarrow$ *by Corollary 2*

$$P\left\{L^N(T) \geq \alpha\right\} \sim \int \Phi\left(\frac{N\,L(T;\psi) - \alpha}{\sqrt{N}\sqrt{V(T;\psi)}}\right) df_\Psi(\psi)$$

with $f_\Psi(\cdot)$ the Gamma-density of Ψ.

Figure 4: β=1.5, γ=2.1, γ_c=1/$tanh(\beta)$≈1.105

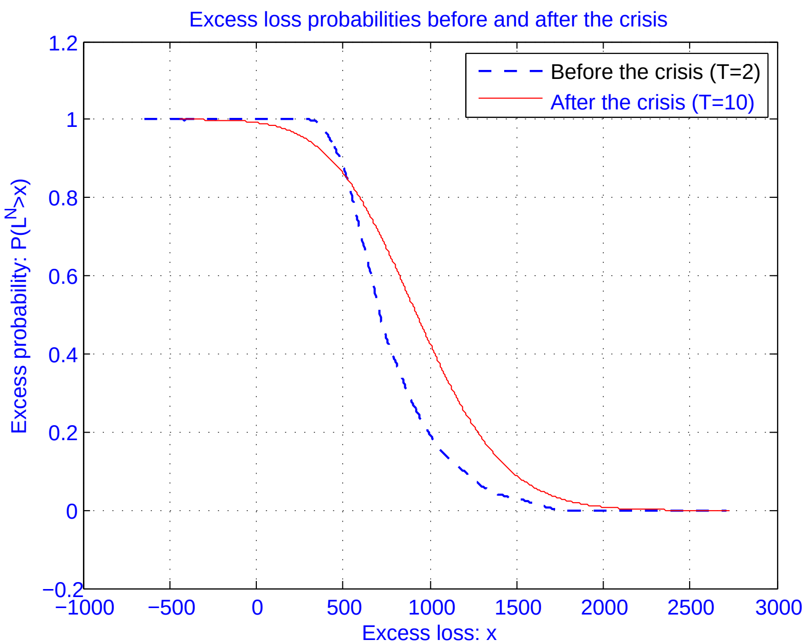


Figure 5: $\gamma_c = 1/tanh(1.5) \approx 1.105$; $\gamma_c = 1/tanh(0.9) \approx 1.396$