Optimal Dividends in Presence of Downside Risk

(joint work with Luis H. R. Alvarez E.)

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Outline

Dividend Payout Problem

- Economic problem: in what way should a firm pay out dividends in order to maximize the expected present value of future dividends to shareholders?
- Mathematical problem: to determine the optimal control policy for a stochastically fluctuating process.
- Answers or at least partial answers are well known in cases when the underlying process is a *linear diffusion* or a *Lévy process* (arithmetic or exponential).
- But what about other jump diffusions?

Downside Risk

- We consider *spectrally negative* jump diffusions, i.e. processes which have only downward jumps but increase continuously. These downward discontinuities represent the *downside risk*.
- Motivation for this model is twofold:
  - The markets tend to react to bad news more dramatically than to good news.
  - Principle of prudence: it is prudent to take into account the potential adverse events (say, instantaneous drops in asset value) and disregard uncertain future profits.

Our Goal

- We shall state reasonably general sufficient conditions for the optimal singular stochastic dividend control to be a *barrier strategy* (except for a potential initial lump sum dividend at time 0).
- We will extend the representation of the value function in terms of the minimal increasing r-excessive map (known in linear diffusion case) to our setup.
- This result implies similar results and representations for the associated optimal impulse control (optimality of a target–trigger policy) and optimal stopping problems (optimality of a single threshold rule).

Underlying Lévy Diffusion X

- The reservoir of assets from which dividends are paid out evolves on $I := (0, \infty)$ according to

$$dX_{t-} = \mu(X_{t-})dt + \sigma(X_{t-})dW_t - \int_{(0,1)} X_{t-} z \tilde{N}(dt, dz), \quad (1)$$

  $X_0 = x > 0$, where $\tilde{N}(dt, dz)$ is a compensated Poisson point process with characteristic measure $\nu = \lambda \mathfrak{m}$, and jump size distribution $\mathfrak{m}$ has a continous density.
- $\mu \in C^1$ and $\sigma > 0$ are assumed to satisfy the usual conditions for the existence of a strong solution.

Assumptions on X

- The absence of speculative bubbles condition

$$\mathbb{E}_x \int_0^{\infty} e^{-rs} X_s ds < \infty, \tag{2}$$

  where $r > 0$ is the discount rate, is met.
- The boundaries 0 and ∞ are natural for X, i.e. unattainable in finite time.
- X is *regular* in the sense that for all $x, y \in I$ it holds that $\mathbb{P}_x(\tau_y < \infty) = 1$, where $\tau_y = \inf\{t > 0 : X_t \geq y\}$.

Infinitesimal Generator of X

- Operator coinciding with the infinitesimal generator of X is defined for f sufficiently smooth by

$$\begin{aligned}(\mathcal{G}f)(x) &= \tfrac{1}{2}\sigma^2(x)f''(x) + \mu(x)f'(x) + \\ &+\lambda \int_{(0,1)} \{f(x - xz) - f(x) + xzf'(x)\}\mathfrak{m}(dz).\end{aligned} \tag{3}$$

- We *assume* that there exists an increasing C^2 solution ψ of $\mathcal{G}_r\psi := \mathcal{G}\psi - r\psi = 0$ such that $\psi(0) = 0$.

Associated Continuous Diffusion $\tilde{X}$

- We define an associated diffusion $\tilde{X}$ by

$$d\tilde{X}_t = \tilde{\mu}(\tilde{X}_t)dt + \sigma(\tilde{X}_t)dW_t, \tag{4}$$

where $\tilde{\mu}(x) = \mu(x) + \lambda x \cdot \int_{(0,1)} z\mathfrak{m}(dz) = \mu(x) + \lambda \bar{z} x$.

Controlled Dynamics

- The controlled cash flow dynamics X_t^D are characterized by the stochastic differential equation

$$dX_t^D = \mu(X_t^D)dt + \sigma(X_t^D)dW_t - \int_{(0,1)} X_t^D z\tilde{N}(dt, dz) - dD_t, \tag{5}$$

  $X_{0-}^D = x$, where D denotes the implemented dividend policy.
- A dividend payout strategy is *admissible* if it is non-negative, adapted, cádlág, and non-decreasing; the class of admissible policies is denoted by $\mathcal{A}$.

Cash Flow Management Problem

- Objective is to solve the singular stochastic control problem

$$V_S(x) = \sup_{D \in \mathcal{A}} \mathbb{E}_x \int_0^{\tau_0^D} e^{-rs} dD_s, \tag{6}$$

where $\tau_0^D = \inf\{t > 0 : X_t^D \leq 0\}$ denotes the lifetime of X^D.

- It is worth emphasizing that in our model liquidation is always the result of a control action (and, thus, *endogenous*), as the assumed boundary behavior of X implies that exogenous liquidation in finite time is not possible.

Net Appreciation Rate

- Define the net appreciation rate $\rho : I \to \mathbb{R}$ of the stock X as $\rho(x) = \mu(x) - rx$ and assume throughout that it has a finite expected cumulative present value.
- This mapping plays a key role in the determination of the optimal payout policy and its value.

Auxiliary Mappings

- define the C^1 mappings $H : I^2 \mapsto \mathbb{R}$ as

$$H(x,y) = \begin{cases} x - y + \frac{\psi(y)}{\psi'(y)} & x \geq y \\ \frac{\psi(x)}{\psi'(y)} & x < y. \end{cases} \tag{7}$$

- For a given fixed $y \in I$ the function $x \mapsto H(x,y)$ satisfies the variational equalities

$$\begin{aligned} (\mathcal{G}_r H)(x,y) &= 0, \quad x < y \\ \partial_x H(x,y) &= 1, \quad x \geq y. \end{aligned}$$

A Crucial Uniqueness and Existence Result (Theorem 1)

Theorem

Assume that the net appreciation rate $\rho(x)$ *satisfies the limiting inequalities* $\lim_{x\to\infty}\rho(x) < 0 \leq \lim_{x\downarrow 0}\rho(x)$, *that there exists a unique threshold* $\hat{x} \in I$ *such that* $\rho(x)$ *is increasing on* $(0, \hat{x})$ *and decreasing on* $(\hat{x}, \infty)$, *and that* $\rho(x)$ *is concave on* $(\hat{x}, \infty)$. *Then equation* $\psi''(x) = 0$ *has a unique root* $x^* \in (\hat{x}, \infty)$ *so that* $\psi''(x) \lesseqqgtr 0$ *for* $x \lesseqqgtr x^*$ *and* $x^* = argmin\{\psi'(x)\}$.

Sketch of Proof (Existence)

- To prove existence, first establish local concavity of $\psi(x)$ near the origin, then show that it cannot become convex on $(0, \hat{x})$ and finally that it has to become convex before $x_0 = \rho^{-1}(0)$.
- To do this by contradiction, use the auxiliary quantity

$$I(x) = r(\psi(x) - x\psi'(x)) - \rho(x)\psi'(x) - J(x, \psi(x)), \tag{8}$$

where $I(x) = \frac{1}{2}\sigma^2(x)\psi''(x)$, and

$$J(x, \psi(x)) = \int_{(0,1)} \{\psi(x - xz) - \psi(x) + xz\psi'(x)\}\nu(dz). \tag{9}$$

Sketch of Proof (Uniqueness)

- To establish uniqueness, consider the derivative of

$$\tilde{I}(x) = (r+\lambda)\left(\frac{\psi(x)}{S'(x)} - x\frac{\psi'(x)}{S'(x)}\right) - \tilde{\rho}(x)\frac{\psi'(x)}{S'(x)} - \tilde{J}(x), \quad (10)$$

where $\tilde{I}(x) = \frac{\sigma^2(x)\psi''(x)}{2S'(x)}$, $\tilde{\rho}(x) = \rho(x) - \lambda x(1-\bar{z})$, $S'(x) = \exp\left(-\int \frac{2\tilde{\mu}(x)dx}{\sigma^2(x)}\right)$ denotes the scale density of the associated diffusion $\tilde{X}$, and

$$\tilde{J}(x) = \int_{(0,1)} \frac{\psi(x(1-z))}{S'(x)}\nu(dz).$$

- Using concavity of $\rho(x)$, the fact that $\tilde{I}'(x^*) > 0$ and Leibniz rule, show that once positive, $\tilde{I}'(x)$ cannot turn negative on (x^*, ∞).

A Superharmonicity Theorem

Theorem

Suppose that the assumptions of Theorem 1 are satisfied and define the function $F : I \mapsto \mathbb{R}_+$ *as* $F(x) = H(x, x^*)$. *Then,*

(A) $F \in C^2(I)$, $(\mathcal{G}_r F)(x) \leq 0$, $F'(x) \geq 1$, *and* $F''(x) \leq 0$ *for all* $x \in I$, *and*

(B) $F(x) \geq H(x, y)$ *and* $F'(x) \geq H_x(x, y)$ *for all* $x, y \in I^2$ *and* $H_y(x, y) < 0$ *for all* $(x, y) \in \mathbb{R}_+ \times (x^*, \infty)$.

Sketch of Proof

- (A): use properties of $(\mathcal{G}_r F)(x)$ and its derivative together with the strict concavity of $\psi(x)$ on $(0, x^*)$.
- (B) follows from known results by Alvarez and Virtanen.

Optimal Singular Control of Dividends

Theorem

Assume that the assumptions of Theorem 1 are satisfied. Then the value of the singular control problem is given by $V_S(x) = H(x, x^*)$. *The value is twice continuously differentiable, monotonically increasing and concave. Moreover, the marginal value (Tobin's marginal q) of the singular control reads as*

$$V_S'(x) = \psi'(x) \sup_{y \geq x} \left\{ \frac{1}{\psi'(y)} \right\} = \begin{cases} 1 & x \geq x^* \\ \frac{\psi'(x)}{\psi'(x^*)} & x < x^*. \end{cases} \tag{11}$$

The corresponding optimal singular control consists of an initial impulse $\xi_{0-} = (x - x^*)^+$ *and a barrier strategy where retained earnings in excess of* x^* *are instantaneously paid out as dividends.*

Sketch of Proof

- Take any $D \in \mathcal{A}$, apply the generalized Itô formula to $(t, x) \mapsto e^{-rt}H(x, x^*)$ for a suitable sequence of increasing stopping times and use the superharmonicity theorem and monotone convergence to establish that the proposed value function dominates the value obtained by strategy D.
- Show that the proposed strategy is admissible.

Further Results

- It can be shown that the obtained representation of the value of the singular control problem implies that also the associated impulse control problem

$$V_{\mathrm{I}}^{c}(x) = \sup_{(\tau,\xi)\in\mathcal{V}} J^{\tau,\xi}(x) = \mathbb{E}_x\left[\sum_{i=1}^{N} e^{-r\hat{\tau}(i)}(\hat{\xi}(i) - c)\right]$$

as well as the associated optimal stopping problems

$$V_{\mathrm{OSP}}(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left[e^{-r\tau}X_\tau\right] \tag{12}$$

and

$$V_{\mathrm{OSP}}^{c}(x) = \sup_{\tau\in\mathcal{T}} \mathbb{E}_x\left[e^{-r\tau}(X_\tau - c)\right], \tag{13}$$

where $\mathcal{T}$ is the set of all $\mathbb{F}$-stopping times, are solvable in terms of the minimal increasing r-excessive map.

References

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