Optimal dividend and reinvestment policies when payments are subject to both fixed and proportional costs

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by

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Models and assumptions

Income process without payments

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

Standing assumptions:

A1. $|\mu(y)| + |\sigma(y)| \leq K(1+y)$ for all $y \geq 0$ and some $K > 0$.

A2. μ and σ are continuously differentiable and the derivatives μ' and σ' are Lipschitz continuous for all $y \geq 0$.

A3. $\sigma^2(y) > 0$ for all $y \geq 0$.

A4. $\mu'(y) \leq r$ for all $y \geq 0$. Here r is a discount factor.

Let

$$Lg(y) = \frac{1}{2}\sigma^2(y)g''(y) + \mu(y)g'(y) - rg(y).$$

Comments on Assumption A4

A4: $\mu'(y) \leq r$ for all $y \geq 0$. Here r is a discount factor.

Consider the special case

$$dX_t = (\mu_0 + \mu_1 X_t)dt + \sigma(X_t)dW_t, \qquad X_0 = x.$$

Here $\mu'(x) = \mu_1$ and furthermore

$$E^x[e^{-rt}X_t] = \left(x + \frac{\mu_0}{\mu_1}\right) e^{(\mu_1 - r)t} - \frac{\mu_0}{\mu_1} e^{-rt}.$$

If $\mu_1 \leq r$ this stabilizes, but if $\mu_1 > r$ it grows to infinity and therefore it is clearly better to wait. The right quantities to compare are therefore $\mu'(x)$ and r, one representing the geometric growth rate and the other the geometric discounting rate. The condition $\mu'(x) \leq r$ just says that in no state should growth rate exceed discounting rate.

The problem

Total dividends paid up to time t is D_t. When reserves hit zero reinvestments are made, total reinvestments up to time t is C_t. Both C and D are nondecreasing and RCLL. Associated costs are

$$\begin{aligned} d\bar{C}_t &= c_0 1_{\{\triangle C_t > 0\}} + c_1 dC_t, \quad 0 \leq c_1 \leq 1, \\ d\bar{D}_t &= d_0 1_{\{\triangle D_t > 0\}} + d_1 dD_t, \end{aligned}$$

where c_0, c_1, d_0 and d_1 all are nonnegative constants.
Therefore

$$\begin{aligned} dY_t = &\ \mu(Y_t)dt + \sigma(Y_t)dW_t + (1-c_1)dC_t - (1+d_1)dD_t \\ &-c_0 1_{\{\triangle C_t > 0\}} - d_0 1_{\{\triangle D_t > 0\}}, \end{aligned}$$

with $Y_{0-} = y$.

For given (C, D) let

$$V_{C,D}(y) = \limsup_{n \to \infty} E^y \left[\int_{0-}^{\nu_n -} e^{-rt} dA_t \right],$$

where $A = D - C$ and $\nu_n = \inf\{t : C_t \vee D_t > n\}$.
We want to find

$$V^*(y) = \sup_{(C,D)} V_{C,D}(y).$$

and also, if it exists, the optimal policy (C^*, D^*).

Shreve, Lehoczky and Gaver (1984).
Same model as here, but without fixed costs.
Richard (1977), Constantinides and Richard (1978), Harrison, Sellke and Taylor (1983).
With fixed costs, but only linear Brownian motion.
Avram, Palmowski and Pistorius (2007).
Spectrally negative Lévy process, but no fixed costs.
Porteus (1977).
Discrete time

Papers with absorbtion at zero

Paulsen (2007).
Same model and expenses as in this paper
Jeanblanc-Picqué and Shiryaev (1995).
Linear Brownian motion.
Shreve, Lehoczky and Gaver (1984).
Same model as here, but without fixed costs.

Papers written for combinations av dividend payments, investment policies and reinsurance policy, but restricted to Brownian motion are

Cadenillas, Sarkar and Zapatero (2007), Cadenillas, Choulli, Taksar and Zhang (2006).

Solution of the problem Consider the variational problem for unknown V, y^*, $\gamma^* \in (0, y^*)$ and $\delta^* \in (0, y^*)$,

$$\begin{aligned}
LV(y) &= 0, \quad 0 < y < y^*, \\
V(\gamma^*) &= V(0) + \frac{\gamma^* + c_0}{1 - c_1}, \\
V'(\gamma^*) &= \frac{1}{1 - c_1}, \\
V(y^*) &= V(y^* - \delta^*) + \frac{\delta^* - d_0}{1 + d_1}, \\
V'(y^* - \delta^*) &= \frac{1}{1 + d_1}, \\
V'(y^*) &= \frac{1}{1 + d_1}, \\
V(y) &= V(y^*) + \frac{y - y^*}{1 + d_1}, \quad y > y^*.
\end{aligned}$$

a) If this has a solution this solution is unique and

$$V(y) = V^*(y), \quad y \geq 0.$$

The optimal policy is to pay δ^* in dividends whenever $Y_{t-} = y^*$ and to reinvest γ^* whenever $Y_{t-} = 0$.

b) If this has no solution there is no optimal policy, but

$$V^*(y) = \lim_{\bar{y} \to \infty} V_{\bar{y}, \gamma(\bar{y}), \delta(\bar{y})}(y)$$

and this limit exists and is finite for every $y \geq 0$.

Proposition 1

a) Assume there is no optimal solution. Then there exists a solution g_2 of $Lg = 0$ so that

$$\lim_{y\to\infty} g_2(y) = \lim_{y\to\infty} g_2'(y) = 0.$$

Furthermore, for any other independent solution g_1,

$$\lim_{y\to\infty} g_1'(y) = \lim_{y\to\infty} \frac{g_1(y)}{y} = \bar{g}_1$$

for some positive and finite $\bar{g}_1$.

b) Assume that there are two solutions g_1 and g_2 of $Lg = 0$ so that

$$\begin{aligned} \lim_{y\to\infty} g_1'(y) &= \bar{g}_1, \\ \lim_{y\to\infty} g_2(y) &= 0, \end{aligned}$$

where $\bar{g}_1$ is finite and nonzero. Assume in addition that

$$\lim_{y\to\infty} \left(\frac{g_1(y)}{\bar{g}_1} - y \right) > \frac{\mu(0)}{r} - d_0.$$

Then there is no optimal solution.

c) Assume there is a solution g of $Lg = 0$ so that

$$\lim_{y\to\infty} \frac{g(y)}{y} = \infty$$

or equivalently

$$\lim_{y\to\infty} g'(y) = \infty.$$

Then there is an optimal solution.

Example - Linear Brownian Motion
Let the income process without dividends follow

$$dX_t = \mu dt + \sigma dW_t,$$

It is easy to verify that $Lg(y) = 0$ has the independent solutions

$$g_i(y) = e^{\theta_i y}, \quad i = 1, 2,$$

where

$$\begin{aligned} \theta_1 &= \frac{1}{\sigma^2}\left(\sqrt{\mu^2 + 2r\sigma^2} - \mu\right) \\ \theta_2 &= -\frac{1}{\sigma^2}\left(\sqrt{\mu^2 + 2r\sigma^2} + \mu\right). \end{aligned}$$

Clearly $\theta_1 > 0$, hence an optimal solution exists by Proposition 1.c. This is the main result of Harrison & al. (1983).

Proposition 2
Assume there is no optimal policy, and let V be the value function. Consider the equation (in $\bar{\gamma}$).

$$\begin{aligned} V'(\bar{\gamma}) &= \frac{1}{1-c_1}, \\ V(\bar{\gamma}) &= V(0) + \frac{\bar{\gamma}+c_0}{1-c_1}. \end{aligned} \tag{1}$$

Furthermore, with g_1 and g_2 as in Proposition 1, write

$$V(y) = a_1 g_1(y) + a_2 g_2(y).$$

a) We have

$$\lim_{y\to\infty} V'(y) = \frac{1}{1+d_1}.$$

b) If $c_1 + d_1 > 0$ then (1) has a unique solution. Furthermore

$$\begin{aligned} a_1 &= \frac{1}{1+d_1}\frac{1}{\bar{g}_1}, \\ a_2 &= \frac{1}{1-c_1}\frac{1}{g_2'(\bar{\gamma})} - \frac{1}{1+d_1}\frac{1}{\bar{g}_1}\frac{g_1'(\bar{\gamma})}{g_2'(\bar{\gamma})}. \end{aligned}$$

Here $\bar{g}_1 = \lim_{y\to\infty} g_1'(y)$ and $\bar{\gamma}$ is the solution of

$$\begin{aligned} c_0 = {} & \frac{1-c_1}{1+d_1}\frac{1}{\bar{g}_1}(g_1(y) - g_1(0)) \\ & + \left(\frac{1}{g_2'(y)} - \frac{1-c_1}{1+d_1}\frac{1}{\bar{g}_1}\frac{g_1'(y)}{g_2'(y)}\right)(g_2(y) - g_2(0)) - y. \end{aligned}$$

c) If $c_1 = d_1 = 0$ there are two possibilities.

(i) The equation (1) has a unique solution and then a_1, a_2 and $\bar{\gamma}$ are is in part b above.

(ii) The equation (1) has no solution, but

$$
\begin{aligned}
a_1 &= \frac{1}{\bar{g}_1}, \\
a_2 &= \frac{\lim_{y\to\infty}\left(\frac{g_1(y)}{g_1} - y\right) - \frac{g_1'(0)}{\bar{g}_1} - c_0}{g_2(0)}.
\end{aligned}
$$

A financial example

Income process without dividends assumed to be a linear Brownian motion with drift μ and diffusion σ, but money can be invested in risk free assets with return r.

Investment costs are incurred with rate $\alpha(Y_t)$ so that total investment costs have intensity $\alpha(Y_t)Y_t$.

Assume that this consists of a fixed part α_0 and a part that is proportional with the amount invested α_1, i.e.

$$\alpha(y)y = \alpha_0 + \alpha_1 y.$$

This gives

$$dX_t = (\mu_0 + (r - \alpha_1)X_t)dt + \sigma dW_t,$$

where $\mu_0 = \mu - \alpha_0$. Assume that $\mu_0 > 0$ and $0 \leq \alpha_1 < r$. When $\alpha_0 = 0$ and $\alpha_1 = r$, this is Brownian motion.

The generator is

$$Lg(y) = \frac{1}{2}\sigma^2 g''(y) + (\mu_0 + (r - \alpha_1)y)g'(y) - rg(y) = 0.$$

Assume first that $\alpha_1 = 0$. Two solutions are

$$\begin{aligned} g_1(y) &= ry + \mu_0, \\ g_2(y) &= e^{-k(y)} U(1, \tfrac{1}{2}, k(y)), \end{aligned}$$

where

$$\begin{aligned} k(y) &= \frac{r}{\sigma^2}\left(y + \frac{\mu_0}{r}\right)^2, \\ U(a,b,x) &= \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{b-a-1} dt, \quad a > 0. \end{aligned}$$

In this case there is no optimal solution, but if $c_1 = d_1 = 0$,

$$V^*(y) = y + \frac{\mu_0}{r} - \frac{c_0}{U(1, \frac{1}{2}, k(0))} e^{-(k(y)-k(0))} U(1, \tfrac{1}{2}, k(y)).$$

The first two terms are the value if there were no costs when reaching zero, i.e. when $c_0 = 0$.

When $\alpha_1 > 0$, we have the solutions

$$\begin{aligned} g_1(y) &= e^{-k(y)} F(1, \tfrac{1}{2}, k(y)), \\ g_2(y) &= e^{-k(y)} U(1, \tfrac{1}{2}, k(y)). \end{aligned}$$

Also

$$e^{-k(y)} F(a, b, k(y)) \sim \left(y + \frac{\mu_0}{r - \alpha_1}\right)^{\frac{r}{r-\alpha_1}},$$

hence there is always a solution.

In all tables fixed values are $\sigma^2 = \mu_0 = 1$, $c_0 = d_0 = 0.1$, $c_1 = d_1 = 0.05$, $r = 0.1$ and $\alpha = 0.02$.

Solutions were obtained by using Runge-Kutta for $g_1(0) = 0$, $g_1'(0) = 1$ and $g_2(0) = 1$, $g_2'(0) = 0$, together with the MATLAB function fsolve.

c_0	0	0.1	1	3	5	7.76	10
y^*	4.50	5.14	5.89	6.33	6.54	6.73	6.84
γ^*	0	0.61	1.31	1.72	1.92	3.10	2.20
$y^* - \delta^*$	0.47	1.06	1.75	2.15	2.35	2.52	2.62
$V^*(0)$	8.81	8.52	7.36	5.13	2.96	0	-2.39
$V^*(1)$	9.77	9.66	9.44	9.15	8.90	8.56	8.29
$V^*(5)$	13.50	13.28	13.23	13.16	13.11	13.08	13.07

d_0	0	0.1	1	3	5	10
y^*	1.94	5.14	14.83	29.28	41.80	70.53
γ^*	0.67	0.61	0.50	0.45	0.43	0.40
$y^* - \delta^*$	1.94	1.06	0.73	0.61	0.57	0.52
$V^*(0)$	8.95	8.52	7.53	6.67	6.19	5.48
$V^*(1)$	10.10	9.66	8.64	7.75	7.26	6.51
$V^*(5)$	13.92	13.38	11.98	10.76	10.08	9.06

$c_0 = d_0$	0	0.1	1	3	5	5.42	10
y^*	1.30	5.14	15.68	30.84	43.80	46.71	73.28
γ^*	0	0.61	1.15	1.45	1.60	1.62	1.80
$y^* - \delta^*$	1.30	1.06	1.37	1.60	1.73	1.75	1.91
$V^*(0)$	9.24	8.52	6.35	3.22	0.53	0	-5.61
$V^*(1)$	10.22	9.66	8.45	7.32	6.59	6.46	5.28
$V^*(5)$	14.04	13.38	11.88	10.66	10.00	9.88	8.99