# On the Wealth Dynamics of Self-financing Portfolios under Endogeneous Prices


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Joint work with Jesper Pedersen and Klaus Schenk-Hoppé.

# Motivation

**Mathematical Finance**

- Classical continuous time theory
- Price process given
- Option pricing
- Optimal investment

**Economics**

- Supply and demand
- Prices by market clearing
- Interaction of investors

- Evolution of investors' wealth
- Price formation
- Optimal strategies

# Classical continuous-time finance

- Investors are price-takers
- Trades have no impact on the market
- Dynamics of asset prices are given by a stochastic process, e.g.

$$S_t = S_0 \exp(\mu t + \sigma B_t).$$

- There is infinite supply of financial assets
- There is infinite divisibility of financial assets

**Standing assumption**

Small investors!!!

- Infinite divisibility of financial assets $\Longrightarrow$ big investors

# Large trader and large trades

1. Option hedging has significant impact on stock prices
   - Empirical "proofs"
   - Large trader models: Frey (1998), Platen and Schweizer (1998), Bank and Baum (2004)
2. Large trades cannot be performed without being noticed
   - splitting large trades into smaller to lower market impact – algorithmic trading
   - using strategies based on econometric and mathematical reasoning: Keym and Madhavan (1996), He and Mamaysky (2005)
   - strategies based on analysis of limit order books

**Limitations**

- only one large trader
- trader's impact on the market is ad-hoc specified

# Equilibrium with heterogeneous agents

- many investors, heterogeneous beliefs
- dividends
- investors are utility maximizers
- prices determined to clear the market
- one-period models and overlapping generations (De Long, Shleifer, Summers, Waldmann)
- dynamic models are very complicated and often unsolvable (Hommes)

# The Market

**Asset $k$** $k = 1, 2$

Price $S_k(t)$

Cumulative dividends $D_k(t)$

$$D_k(t) = \int_0^t \delta_k(s)ds$$

Assets in net supply of 1.

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Assets in net supply of 1.

**Investor $i$** $\quad i = 1, 2$

Wealth $V^i(t)$

Consumption rate $cV^i(t)$

Constant proportions trading strategy $(\lambda_1^i, \lambda_2^i)$

Portfolio

number of shares of asset $k$:

$$\frac{\lambda_k^i V^i(t)}{S_k(t)}$$

# Wealth dynamics

$$dV^i(t) = \text{capital gains} + \text{dividends} - \text{consumption}$$

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**Consumption**

$$cV^i(t)dt$$

# Wealth dynamics

$$dV^i(t) = \text{capital gains} + \text{dividends} - \text{consumption}$$

$$dV^i(t) = \sum_{k=1}^{2} \frac{\lambda_k^i V^i(t)}{S_k(t)} \Big( dS_k(t) + dD_k(t) \Big) - cV^i(t)dt$$

# Market clearing

**Market clearing condition**

$$\frac{\lambda_k^1 V^i(t)}{S_k(t)} + \frac{\lambda_k^2 V^i(t)}{S_k(t)} = 1, \qquad k = 1, 2.$$

Equivalent to the net clearing condition:

$$d\theta_k^1(t) + d\theta_k^2(t) = 0, \qquad k = 1, 2.$$

# Price formation

Dividend intensities $\delta_k(t)$

$+$

Investment strategies $(\lambda_1^i, \lambda_2^i)$

$+$

Investor's wealth dynamics

$+$

Market clearing condition

$$\Downarrow$$

Asset prices $S_k(t)$, $k = 1, 2$

# Price formation

**Theorem**

1. *For any feasible* $\big(V^1(0), V^2(0)\big)$ *there exists a unique* $\big(V^1(t), V^2(t)\big)$ *satisfying wealth dynamics and market clearing condition.*
2. *Asset price dynamics are given by*

$$S_k(t) = \lambda_k^1 V^1(t) + \lambda_k^2 V^2(t), \qquad k = 1, 2.$$

# Markovian dividend intensities

Relative dividend intensity $$\rho(t) = \frac{\delta_1(t)}{\delta_1(t) + \delta_2(t)} \in [0,1]$$

**Assumptions**

1. $\rho(t)$ is a positively recurrent Markov process
2. its state space is countable
3. its initial distribution is stationary (stationary economy)

**Theorem**

*Relative dividend intensity process is ergodic:*

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \rho(s) ds = \mathbb{E}\rho(0).$$

# Selection dynamics

## Theorem

*If investor 1 follows strategy*

$$\Pi^* = (\lambda_1^1, \lambda_2^1) = (\mathbb{E}\rho(0), 1 - \mathbb{E}\rho(0))$$

*and investor 2 follows a strategy* $(\lambda_1^2, \lambda_2^2) \neq \Pi^*$ *then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{V^1(s)}{V^1(s) + V^2(s)}\, ds = 1.$$

## Remarks

1. $\Pi^*$ is based on fundamental valuation.
2. Relative wealth of investor 2 converges to zero.
3. At odds with findings in discrete-time evolutionary models (Evstigneev, Hens, Schenk-Hoppé).

# Price dynamics

If one of the investors follows trading strategy $\Pi^*$ then asset prices converge:

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{S_1(s)}{S_1(s) + S_2(s)} ds = \mathbb{E}\rho(0).$$

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**Fundamental valuation**

$$\frac{\mathbb{E}\delta_1(0)}{\mathbb{E}\delta_1(0)+\mathbb{E}\delta_2(0)}$$

**Our valuation**

$$\mathbb{E}\left(\frac{\delta_1(0)}{\delta_1(0)+\delta_2(0)}\right)$$

**Remarks**

1. Fundamental valuation comes as a result of computing average historical payoffs.
2. Our valuation is a fundamentally different benchmark.

# Almost sure convergence

**Assumption**

For every state $x$

$$\mathbb{E}^x(\tau_x)^2 < \infty.$$

**Theorem**

1. *If investor 1 follows strategy $\Pi^*$ and investor 2 follows a strategy $(\lambda_1^2, \lambda_2^2) \neq \Pi^*$ then*

$$\lim_{t\to\infty} \frac{V^1(t)}{V^1(t) + V^2(t)} = 1 \quad a.s.$$

2. *If one of the investors follows strategy $\Pi^*$ then asset prices converge to our benchmark value:*

$$\lim_{t\to\infty} \frac{S_1(t)}{S_1(t) + S_2(t)} = \mathbb{E}\rho(0) \quad a.s.$$

# Proof

## What we hoped to do

- Linearization and Lagrange multipliers
- Multiplicative Ergodic Theorem

Why? It works fine in discrete-time.

- Continous-time setting supprised us. Lagrange multiplier at the steady state is zero!

## What we have done

- Domination by a Ricatti-type equation with random coefficients.
- One coefficient depending on the solution of the original problem.
- Arcsine law.

# Summary

- Heterogeneous investors in continuous time model
- Wealth dynamics
- Optimal investment strategies
- Asset pricing - new valuation benchmark

- Open problems
  - Time varying investment strategies
  - More agents