Portfolio optimization with transaction costs

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Outline

The Merton problem with transaction costs

- **Goal**: Maximize expected utility from consumption

$$\mathbb{E}\left(\int_0^\infty e^{-\delta t} u(c_t) dt\right)$$

- Here: $u(x) = \log(x)$
- c admissible consumption rate (no debts)
- Bank account: 1 (no interest paid)
- Stock price S: Modeled as geometric Brownian motion
- Proportional transaction costs μ, λ (e.g. 1%)

The Merton problem with transaction costs

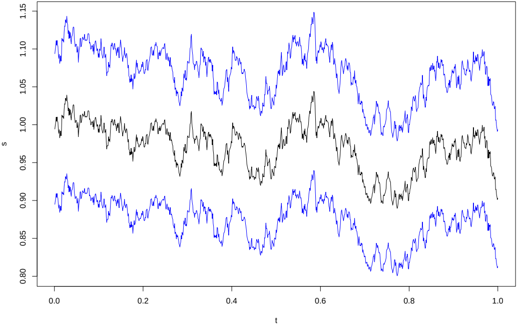
Without transaction costs (Merton [1971]):

- Fixed fraction of wealth in stock (e.g. 31%)
- Consumption rate is fixed proportion of wealth
- Both numbers explicitly known

With transaction costs (Magill and Constantinides [1976], Davis and Norman [1990]):

- Fraction of wealth in stock in fixed corridor (e.g. 20-40%)
- Consumption rate is function of wealth in cash and stock
- Corridor known only as solution to free boundary problem

A general principle
Shadow prices



Optimal portfolio with transaction costs?

A general principle

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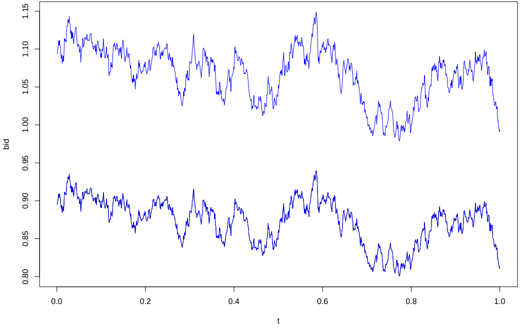

Optimal portfolio with transaction costs?

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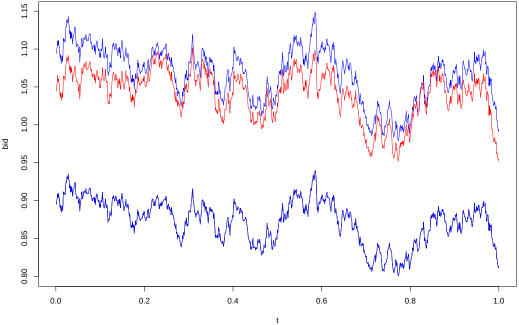

Optimal portfolio with transaction costs

$$\Longleftrightarrow$$

Optimal portfolio without transaction costs for shadow price

A general principle

Shadow prices

- **Idea**: Problem with transaction costs as problem without transaction costs for different price process
- Shadow price at boundary when optimal strategy transacts

Appearances in various fields:

- Jouini and Kallal [1995]: No-arbitrage
- Lamberton et al. [1998]: Local risk minimization
- Cvitanić and Karatzas [1996], Loewenstein [2000]: Portfolio optimization

Useful for computations?

Application to Merton problem with transaction costs

Real price processes:

- **Stock price**(discounted): $dS_t/S_t = \alpha d_t + \sigma dW_t$
- **Bid price**: $(1-\mu)S_t$
- **Ask price**: $(1+\lambda)S_t$

Shadow price process $\widetilde{S} \in [(1-\mu)S, (1+\lambda)S]$:

- $\widetilde{S}_t = \exp(C_t)S_t$
- $C_t = \log(\widetilde{S}_t/S_t)$ deviation from real price
- $C_t \in [\log(1-\mu), \log(1+\lambda)]$

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Dynamics of C?

Application to Merton problem with transaction costs

Ansatz:

- Itô process $dC_t = \widetilde{\alpha}(C_t)dt + \widetilde{\sigma}(C_t)dW_t$

$$\Rightarrow d\widetilde{S}_t/\widetilde{S}_t = \text{Drift}(C_t)d_t + \text{Diffusion}(C_t)dW_t$$

Optimal strategy (without transaction costs):

- Consumption: $\delta\widetilde{V}_t$
- Fraction of stocks: $\pi(C_t) = \frac{\text{Drift}(C_t)}{\text{Diffusion}(C_t)^2}$
- Use transformation $\frac{1}{1+\exp(f(C_t))} = \pi(C_t)$

$\Rightarrow$ Need to determine 3 functions: $\widetilde{\alpha}$, $\widetilde{\sigma}$, f
$\Rightarrow$ $f(\log(1-\mu))$, $f(\log(1+\lambda))$ determine corridor

Application to Merton problem with transaction costs

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Application to Merton problem with transaction costs

- **Optimality**:

$$\frac{1}{1+\exp(-f)} = \frac{\text{Drift}}{\text{Diffusion}^2} \tag{I}$$

- **No trading within bounds**: $d\varphi_t = 0$ for optimal φ
- Itô's formula:

$$\begin{aligned} d\varphi_t = {} & \text{somefunction}(f, f', f'', \widetilde{\alpha}, \widetilde{\sigma})dt \\ & + \text{anotherfunction}(f, f', \widetilde{\alpha}, \widetilde{\sigma})dW_t \end{aligned}$$

- Hence

$$0 = \text{somefunction}, \tag{II}$$
$$0 = \text{anotherfunction} \tag{III}$$

- 3 conditions

Application to Merton problem with transaction costs

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Solution to Equations I-III:

$$
\begin{aligned}
\tilde{\sigma} &= \frac{\sigma}{f' - 1} \\
\tilde{\alpha} &= -\alpha + \sigma^2 \left(\frac{f'}{f' - 1} \right) \left(\frac{1}{1 + e^{-f}} \right)
\end{aligned}
$$

f satisfies the ODE

$$
\begin{aligned}
f''(x) &= \left(\tfrac{2\delta}{\sigma^2}(1 + e^{f(x)}) \right) + \left(\tfrac{2\alpha}{\sigma^2} - 1 - \tfrac{4\delta}{\sigma^2}(1 + e^{f(x)}) \right) f'(x) \\
&\quad + \left(\tfrac{4\alpha}{\sigma^2} + 2 - \tfrac{2\delta}{\sigma^2}(1 + e^{f(x)}) + \tfrac{1 - e^{-f(x)}}{1 + e^{-f(x)}} \right) (f'(x))^2 \\
&\quad + \left(\tfrac{2\alpha}{\sigma^2} + \tfrac{2}{1 + e^{-f(x)}} \right) (f'(x))^3
\end{aligned}
$$

Still missing:
Boundary conditions for $x = \log(1 - \mu)$ and $x = \log(1 + \lambda)$

Application to Merton problem with transaction costs

Heuristics for boundary conditions:

- Optimal fraction $\pi(C_t)$: Reflected diffusion (e.g. between 20% and 40%) $\Rightarrow$ local time at boundary
- Hence $f(C_t) = \log\left(\frac{1-\pi(C_t)}{\pi(C_t)}\right)$ has local time
- Our Ansatz: $\tilde{S}_t$ (and hence C_t) Itô process, i.e. no local time at boundary
- Intuition: otherwise infinite position optimal at boundary

Consequence: Contradiction unless $f' = \infty$ on boundary

- Boundary conditions

$$\begin{aligned} f'(\log(1-\mu)) &= \infty \\ f'(\log(1+\lambda)) &= \infty \end{aligned}$$

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Application to Merton problem with transaction costs

Numerical solution: Consider $g = f^{-1}$

- ODE for g:

$$\begin{aligned} g''(y) &= \left(\tfrac{1-e^{-y}}{1+e^{-y}} + 1 - \tfrac{2\alpha}{\sigma^2}\right) \\ &\quad + \left(\tfrac{4\alpha}{\sigma^2} - 2 - \tfrac{1-e^{-y}}{1+e^{-y}} - \tfrac{2\delta}{\sigma^2}(1+e^y)\right) g'(y) \\ &\quad + \left(-\tfrac{2\alpha}{\sigma^2} + 1 - \tfrac{4\delta}{\sigma^2}(1+e^y)\right)(g'(y))^2 \\ &\quad - \left(\tfrac{2\delta}{\sigma^2}(1+e^y)\right)(g'(y))^3 \end{aligned}$$

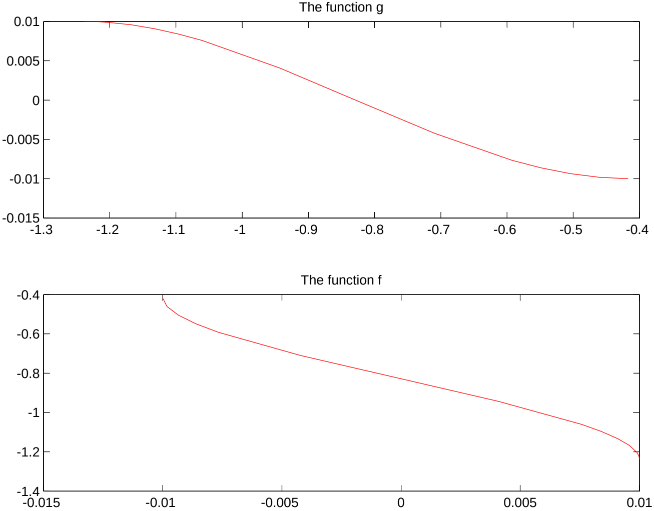
- Free boundary: y_1, y_2 with

$$\begin{aligned} g(y_1) &= \log(1-\mu), \quad g'(y_1) = 0 \\ g(y_2) &= \log(1+\lambda), \quad g'(y_2) = 0 \end{aligned}$$

- Free boundaries y_1, y_2 determine corridor, $g = f^{-1}$ determines dynamics of C and hence $\widetilde{S} = \exp(C)S$

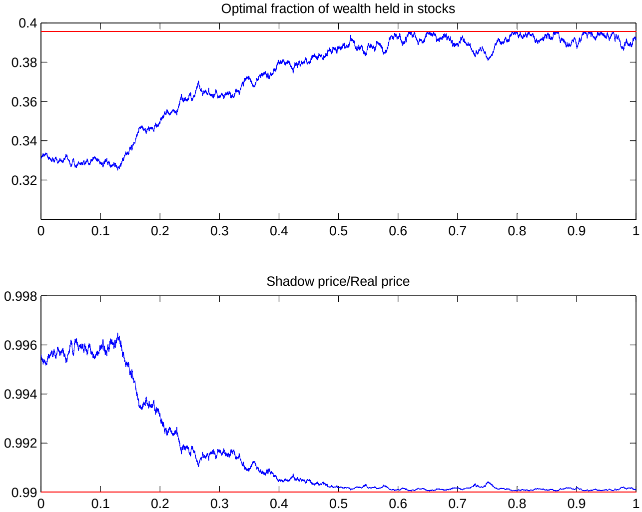
Application to Merton problem with transaction costs

Numerical solution ct'd



Application to Merton problem with transaction costs

Simulation



Application to Merton problem with transaction costs
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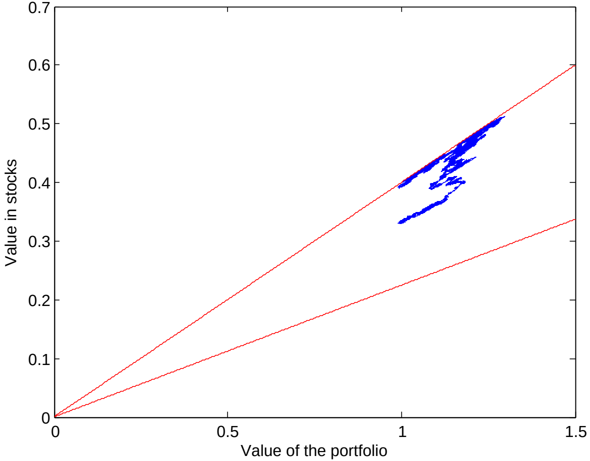

Summary

Computation of conditions:

1. Optimality without transaction costs,
2. Constant trading strategy within bounds,
3. Boundary conditions via Itô process assumption.

Verification:

1. Prove existence of a solution to free boundary problem.
2. Prove existence of corresponding processes $\tilde{S}$ etc.
3. Show that optimal investment in $\tilde{S}$ trades only at boundary.

References

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