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MULTIDIMENSIONAL COHERENT RISK MEASURES

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COHERENT RISK MEASURES

Definition 1.2. (ADEH97) A coherent utility function on L^∞ is a map $u: L^\infty \to \mathbb{R}$ with the properties

(i) (Superadditivity) $u(X+Y) \geq u(X)+u(Y)$;

(ii) (Monotonicity) If $X \leq Y$, then $u(X) \leq u(Y)$;

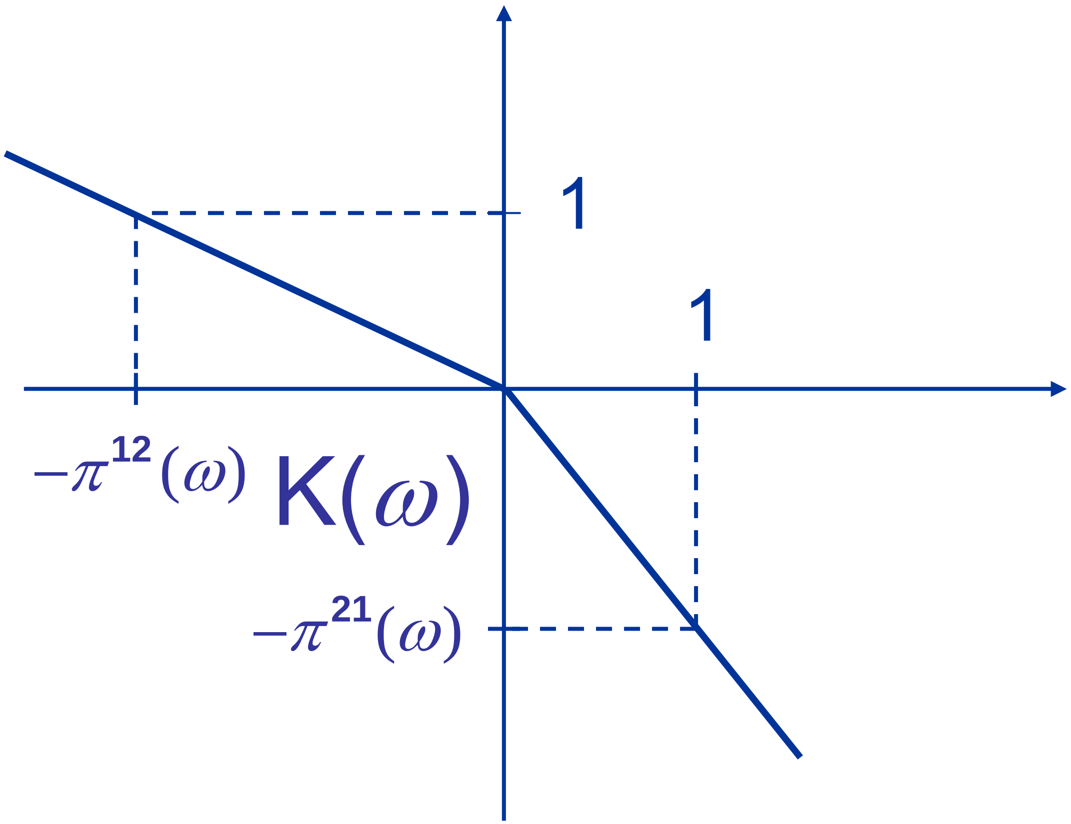
(iii) (Translation invariance) $u(X+m) = u(X)+m$ for all $m \in \mathbb{R}$;

(iv) (Positive homogenuity) $u(\lambda X) = \lambda u(X)$ for $\lambda \geq 0$;

(v) (Fatou property) If $|X_n| \leq 1$, $X_n \xrightarrow{P} X$, then $u(X) \geq \overline{\lim}_n u(X_n)$.

The corresponding coherent risk measure is $\rho(X) = -u(X)$.

PARTIAL ORDERING



$X \prec Y$ if $X(\omega) - Y(\omega) \in K(\omega)$ P-a.s., where $K(\omega)$ is the cone of currency exchange rates.

AXIOMS

Definition 1.1. A multidimensional coherent utility function on $(L^{\infty})^d$ is a map $u: (L^{\infty})^d \to C$, $u \neq \mathbb{R}^d$, $u = u + \mathbb{R}^d_-$ with the properties

(i) (Superadditivity) $u(X+Y) \supseteq u(X) + u(Y)$;

(ii) (Monotonicity) If $X \prec Y$, then $u(X) \subseteq u(Y)$;

(iii) (Translation invariance) $u(X+m) = u(X) + m$ for all $m \in \mathbb{R}^d$;

(iv) (Positive homogenuity) $u(\lambda X) = \lambda u(X)$ for $\lambda > 0$;

(v) (Fatou property) If $\|X_n\| \leq c$, $X_n \xrightarrow{P} X$, then $u(X) \supseteq \overline{\lim}_n u(X_n)$, i.e. if x belongs to infinitely many $u(X_n)$, then x belongs to $u(X)$.

The corresponding multidimensional coherent risk measure is $\rho(X) = -u(X)$.

AXIOMS

Remarks.

(i) If u is a coherent utility function, then $v(X) = (-\infty, u(X)]$ is a multidimensional coherent utility function with $d = 1$, $K(\omega) = \mathbb{R}_-$.

(ii) If u is a multidimensional coherent utility function with $d = 1$, $K(\omega) = \mathbb{R}_-$, then $v(X) = \sup\{x \in \mathbb{R} : x \in u(X)\}$ is a coherent utility function.

REPRESTATION THEOREMS

Theorem 2.1. A function $u:(L^{\infty})^{d} \to C$ is a multidimensional coherent utility function iff there exists nonempty set $D \subseteq (L^{1})^{d}$ such that $Z(\omega) \in K^{*}(\omega)$ P-a.s. and

$$u(X) = \left\{ x \in R^{d} : \forall Z \in D \ \sum_{i=1}^{d} E x^{i} Z^{i} \le \sum_{i=1}^{d} E X^{i} Z^{i} \right\}, \qquad (2.1)$$

where $K^{*}(\omega)$ – polar to $K(\omega)$, i.e.

$$K^{*}(\omega) = \left\{ x \in R^{d} : \forall z \in K(\omega) \ \langle x, z \rangle \le 0 \right\}.$$

Theorem 2.2. (ADEH99) A function $u: L^{\infty} \to \mathbb{R}$ is a coherent utility function iff there exists nonempty set $D \subseteq \mathcal{P}$ such that

$$u(X) = \inf_{Q \in D} E_{Q} X. \qquad (2.2)$$

REPRESENTATION THEOREMS

Definition 2.3. The largest set, for which (2.1) is true, is called the determining set for multidimensional coherent utility function.

Definition 2.4. A multidimensional coherent utility function on $(L^0)^d$ is a map $u:(L^0)^d \to C \cup \{\varnothing\}$ defined as

$$u(X) = \left\{ x \in R^d : \forall Z \in D \ \sum_{i=1}^{d} Ex^i Z^i \le \sum_{i=1}^{d} EX^i Z^i \right\}, \qquad (2.3)$$

where D – set of d-dimensional random vectors $Z \in (L^1)^d$ such that $Z(\omega) \in K^*(\omega)$ P-a.s., and $EX^i Z^i = E(X^i Z^i)^+ - E(X^i Z^i)^-$, with an agreement $(+\infty) - (-\infty) = -\infty$, and if in the sum we have one item equal to $-\infty$, then the sum is equal to $-\infty$.

REPRESENTATION THEOREMS

Remarks.

(i) The above definitions are the multidimensional analogues of the classical ones.

(ii) Clearly, the determining set is a convex cone. If multidimensional coherent utility function is on $(L^{\infty})^d$, then the determining set is $(L^1)^d$-closed.

(iii) If D – $(L^1)^d$-closed convex cone and multidimensional coherent utility function is defined by (2.1) or (2.3), then D is the determining set for u.

EXTREME ELEMENTS

Let $L = \left\{ Z \in (L^1)^d : \sum_{i=1}^{d} EZ^i = 1 \right\}.$

Then we can introduce some important spaces:

$$L^1_w(D) = \left\{ X \in (L^0)^d : \sup_{Z \in D \cap L} \sum_{i=1}^{d} \left| EZ^i X^i \right| < \infty \right\};$$

$$L^1_s(D) = \left\{ X \in (L^0)^d : \sup_{Z \in D \cap L} \sum_{i=1}^{d} \left| EZ^i X^i \right| \mathrm{I}\left\{ \left| X_i \right| > n \right\} > 0 \right\}.$$

EXTREME ELEMENTS

Definition 3.1. Let u be a multidimensional coherent utility function with the determining set D. Let $X \in (L^0)^d$, $x \in \partial u(X)$. We will call a nonzero vector $Z \in D$ an <u>extreme element for X at point x</u> if $\sum_{i=1}^{d} EZ^i X^i = \sum_{i=1}^{d} EZ^i x^i$.

The set of extreme elements for X at point x is denoted by $\mathcal{X}_D(X, x)$.

Proposition 3.2. If $D \cap L$ is weakly compact and $X \in L_s^1(D)$, then $\mathcal{X}_D(X, x) \neq \varnothing$.

EXTREME ELEMENTS

The financial problems, for solution of which we use extreme elements:

(i) Capital allocation;

(ii) Risk contribution.

THANK YOU FOR YOUR ATTENTION !

- Axiomatization of multidimensional coherent risk measures (utility functions) and their connection with coherent risk measures (utility functions) in one-dimensional case.

- Representation of multidimensional coherent utility functions and introducing of the notion determining set in multidimensional case.

- Extreme elements as one of the basic objects for solving some financial problems.