A model for a large investor trading at market indifference prices

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Model for a "small" trader

Input: price process $S = (S_t)$ for traded stock.

Key assumption: trader's actions do not affect S.

For a *simple* strategy with a process of stock *quantities*:

$$Q_t = \sum_{n=1}^{N} \theta_n 1_{(t_{n-1}, t_n]},$$

where $0 = t_0 < \cdots < t_N = T$ and $\theta_n \in \mathbf{L}^0(\mathcal{F}_{t_{n-1}})$, the terminal *value*

$$V_T = V_T(Q) = \sum_{n=1}^{N} \theta_n (S_{t_n} - S_{t_{n-1}})$$

Mathematical challenge: define terminal wealth V_T for *general* $Q = (Q_t)$.

Passage to continuous time trading

Two steps:

1. Establish that S is a *semimartingale*
   1.1 $\Leftrightarrow \exists$ limit of discrete sums, when sequence (Q^n) of simple integrand converges uniformly (Bechteler-Dellacherie)
   1.2 $\Leftarrow$ Absence of arbitrage for *simple strategies* (NFLBR) (Delbaen & Schachermayer (1994)).
2. If S is a semimartingale, then we can extend the map

$$Q \to V_T(Q)$$

from simple to general (predictable) strategies Q arriving to *stochastic integrals*:

$$V_T(Q) = \int_0^T Q_t dS_t.$$

Basic results for the "small" trader model

Fundamental Theorems of Asset Pricing:

1. Absence of arbitrage for general admissible strategies (NFLVR) $\Leftrightarrow$ S is a local martingale under an equivalent probability measure (Delbaen & Schachermayer (1994)).
2. Completeness $\Leftrightarrow$ Uniqueness of a martingale measure for S (Harrison & Pliska (1983), Jacod (1979)).

Arbitrage-free pricing formula: in complete financial model the arbitrage-free price for a European option with maturity T and payoff ψ is given by

$$p = \mathbb{E}^*[\psi],$$

where $\mathbb{P}^*$ is the unique martingale measure.

“Desirable” features of a “large” trader model

Logical requirements:

1. Allow for general continuous-time trading strategies.
2. Obtain the “small” trader model in the limit:

$$V_T(\epsilon Q) = \epsilon \int_0^T Q_t dS_t^{(0)} + o(\epsilon), \quad \epsilon \to 0.$$

Practical goal: computation of *liquidity* or *price impact* corrections to prices of derivatives:

$$p(\epsilon) = \epsilon \mathbb{E}^*[\psi] + \underbrace{\frac{1}{2}\epsilon^2 C(\psi)}_{\text{liquidity correction}} + o(\epsilon^2).$$

Here $p(\epsilon)$ is a “price” for ϵ contingent claims ψ. Of course, we expect to have

$$C(\psi) \leq 0 \text{ for all } \psi \text{ and } < 0 \text{ for some } \psi.$$

Literature (very incomplete!)

Model is an input: Jarrow (1992), (1994);
Frey and Stremme (1997);
Platen and Schweizer (1998);
Papanicolaou and Sircar (1998);
Cuoco and Cvitanic (1998);
Cvitanic and Ma (1996);
Schonbucher and Wilmott (2000);
Cetin, Jarrow and Protter (2002);
Bank and Baum (2003);
Cetin, Jarrow, Protter and Warachka (2006),
...

Model is an output (a result of equilibrium):
Kyle (1985), Back (1990), Gârleanu, Pedersen, Poteshman (1997)...

Financial model

1. Uncertainty and the flow of information are modeled, as usual, by a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.
2. Traded securities are European contingent claims with maturity T and payments $\psi = (\psi^i)$.
3. Prices are quoted by a finite number of market makers.
   3.1 Utility functions $(u_m(x))_{x \in \mathbf{R}, 1 \leq m \leq M}$ (defined on *real line*):

$$\frac{1}{c} < -\frac{u_m'(x)}{u_m''(x)} < c \text{ for some } c > 0.$$

   $\Rightarrow u_m$ has exp-like behavior. In particular, u_m is bounded above and we can assume that

$$u_m(\infty) = 0.$$

   3.2 Initial (random) endowments $\alpha_0 = (\alpha_0^m)_{1 \leq m \leq M}$ ($\mathcal{F}$-measurable random variables) form a *Pareto optimal allocation*.

Pareto allocation

Definition

A vector of random variables $\alpha = (\alpha^m)_{1 \le m \le M}$ is called *a Pareto allocation* if there is no other allocation $\beta = (\beta^m)_{1 \le m \le M}$ of the same total endowment:

$$\sum_{m=1}^{M} \beta^m = \sum_{m=1}^{M} \alpha^m,$$

which would leave all market makers not worse and at least one of them better off in the sense that

$$\mathbb{E}[u_m(\beta^m)] \ge \mathbb{E}[u_m(\alpha^m)] \quad \text{for all } 1 \le m \le M,$$

and

$$\mathbb{E}[u_m(\beta^m)] > \mathbb{E}[u_m(\alpha^m)] \text{ for some } 1 \le m \le M.$$

Pricing measure of Pareto allocation

First-order condition: We have an equivalence between

1. $\alpha = (\alpha^m)_{1 \le m \le M}$ is a Pareto allocation.
2. The ratios of the marginal utilities are non-random:

$$\frac{u_m'(\alpha^m)}{u_n'(\alpha^n)} = \text{const}(m, n).$$

Pricing measure $\mathbb{Q}$ of a Pareto allocation α is defined by the marginal rate of substitution rule:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{u_m'(\alpha^m)}{\mathbb{E}[u_m'(\alpha^m)]}, \quad 1 \le m \le M.$$

(Marginal) price process of traded contingent claims ψ corresponding to a Pareto allocation α is defined to be

$$S_t = \mathbb{E}_{\mathbb{Q}}[\psi | \mathcal{F}_t]$$

A trading of very small quantities at this price does not change the expected utilities of market makers.

Simple strategy

Strategy: a process of quantities $Q = (Q_t)$ of ψ.
Goal: specify the terminal value $V_T = V_T(Q)$.

Consider a *simple* strategy with the process of quantities:

$$Q_t = \sum_{n=1}^{N} \theta_n 1_{(t_{n-1}, t_n]},$$

where θ_n is $\mathcal{F}_{\tau_{n-1}}$-measurable.
We shall define the corresponding *cash balance process*:

$$X_t = \sum_{n=1}^{N} \xi_n 1_{(t_{n-1}, t_n]},$$

where ξ_n is $\mathcal{F}_{\tau_{n-1}}$-measurable.

Trading at initial time

1. The market makers start with the initial Pareto allocation $\alpha_0 = (\alpha_0^m)_{1 \leq m \leq M}$ of the total (random) endowment:

$$\Sigma_0 := \sum_{m=1}^{M} \alpha_0^m.$$

2. After the trade in θ_1 shares at the cost ξ_1, the total endowment becomes

$$\Sigma_1 = \Sigma_0 - \xi_1 - \theta_1 \psi.$$

3. Σ_1 is redistributed as a *Pareto allocation* $\alpha_1 = (\alpha_1^m)_{1 \leq m \leq M}$.
4. **Key condition:** *the expected utilities of market makers do not change*, that is,

$$\mathbb{E}[u_m(\alpha_1^m)] = \mathbb{E}[u_m(\alpha_0^m)], \quad 1 \leq m \leq M.$$

Trading at time t_n

1. The market makers arrive to time t_n with $\mathcal{F}_{t_{n-1}}$-Pareto allocation α_n of the total endowment:

$$\Sigma_n = \Sigma_0 - \xi_n - \theta_n \psi.$$

2. After the trade in $\theta_{n+1} - \theta_n$ shares at the cost $\xi_{n+1} - \xi_n$, the total endowment becomes

$$\begin{aligned} \Sigma_{n+1} =& \Sigma_n - (\xi_{n+1} - \xi_n) - (\theta_{n+1} - \theta_n)\psi \\ =& \Sigma_0 - \xi_{n+1} - \theta_{n+1}\psi. \end{aligned}$$

3. Σ_{n+1} is redistributed as $\mathcal{F}_{t_n}$-*Pareto allocation* α_{n+1}.
4. **Key condition:** *the conditional expected utilities of market makers do not change*, that is,

$$\mathbb{E}[u_m(\alpha^m_{n+1})|\mathcal{F}_{t_n}] = \mathbb{E}[u_m(\alpha^m_n)|\mathcal{F}_{t_n}], \quad 1 \leq m \leq M.$$

Final step

The large trader arrives at maturity $t_N = T$ with

1. quantity $Q_T = \theta_N$ of the traded contingent claims ψ.
2. cash amount $X_T = \xi_N$.

Hence, finally, her terminal wealth is given by

$$V_T := X_T + Q_T \psi.$$

Lemma

For any simple strategy Q the cash balance process $X = X(Q)$ and the terminal wealth $V_T = V_T(Q)$ are well-defined.

Mathematical challenge: define terminal wealth V_T for *general* strategy Q.

More on economic assumptions

The model is essentially based on two economic assumptions:

Market efficiency After each trade the market makers form a *complete Pareto optimal allocation*.
$\Leftrightarrow$ They can trade anything with each other (not only ψ)!

Information The market makers do not anticipate (or can not predict the direction of) future trades of the large economic agent.
$\Leftrightarrow$ Two strategies coinciding on $[0, t]$ and different on $[t, T]$ will produce the same effect on the market up to time t.
$\Leftrightarrow$ The agent can split any order in a sequence of very small trades at marginal prices.
$\Leftrightarrow$ The expected utilities of market makers do not change.

Remark

From the investor's point of view this is the most "friendly" type of interaction with market makers.

Comparison with Arrow-Debreu equilibrium

Economic assumptions behind a large trader model based on Arrow-Debreu equilibrium:

Market efficiency (Same as above)
After re-balance the market makers form a *Pareto optimal allocation*.
$\Leftrightarrow$ They can trade anything between each other (not only ψ)!

Information The market makers have *perfect knowledge* of strategy Q.
$\Leftrightarrow$ Changes in Pareto allocations occur only at initial time.
$\Rightarrow$ Expected utilities of market makers *increase* as the result of trade.

Model based on Arrow-Debreu equilibrium

Given a strategy Q the market makers immediately change the initial Pareto allocation α_0 to another Pareto allocation $\widetilde{\alpha} = \widetilde{\alpha}(Q)$ with pricing measure $\widetilde{\mathbb{P}}$, the price process

$$\widetilde{S}_t := \mathbb{E}_{\widetilde{\mathbb{P}}}[\psi|\mathcal{F}_t]$$

and total endowment

$$\widetilde{\Sigma} := \sum_{m=1}^{M} \widetilde{\alpha}^m$$

such that

$$\Sigma_0 - \widetilde{\Sigma} = \int_0^T Q_t d\widetilde{S}_t,$$

and the following "clearing" conditions hold true:

$$\mathbb{E}_{\widetilde{\mathbb{P}}}[\alpha_0^m] = \mathbb{E}_{\widetilde{\mathbb{P}}}[\widetilde{\alpha}^m], \quad 1 \leq m \leq M.$$

Process of Pareto allocations

Back to our model.

Mathematical challenge: define terminal wealth for *general* Q.

Consider a simple strategy

$$Q_t = \sum_{n=1}^{N} \theta_n 1_{(t_{n-1}, t_n]},$$

where θ_n is $\mathcal{F}_{\tau_{n-1}}$-measurable and denote by

$$A_t = \sum_{n=1}^{N} \alpha_n 1_{(t_{n-1}, t_n]}$$

the corresponding (non-adapted!) process of Pareto allocations.

Remark

The Pareto allocation A_t contains all information at time t but is not $\mathcal{F}_t$-measurable (*infinite-dimensional* sufficient statistic).

Process of indirect utilities

The process of expected (indirect) utilities for market makers:

$$U_t^m = \mathbb{E}[u_m(A_t^m)|\mathcal{F}_t], \quad 0 \le t \le T,\ 1 \le m \le M.$$

Crucial observation: for a simple strategy Q at any time t

$$\text{knowledge of } (U_t, Q_t) \leftrightarrow \text{knowledge of } A_t.$$

$\Rightarrow (U, Q)$ is a *finite-dimensional* (!) sufficient statistic.

Technical assumptions

Assumption

The utility functions of market makers have bounded *prudence* coefficient:

$$\left| -\frac{u'''(x)}{u''(x)} \right| \leq K, \text{ for some constant } K > 0.$$

Assumption

The filtration is generated by a Brownian motion $W = (W^i)$ and the *Malliavin derivatives* of the total initial endowment Σ_0 and the payoffs $\psi = (\psi^k)$ are bounded:

$$|\mathbf{D}_t(\Sigma_0)| + |\mathbf{D}_t(\psi)| < K, \quad 0 \leq t \leq T, \text{ for some constant } K > 0.$$

Passage to continuous-time trading

The key intermediate result is the following

Theorem

Assume the technical conditions above. There is a continuously differentiable stochastic vector field $G = (G_t(u, q))$ *and a constant* $K > 0$ *such that*

$$|G_t^m| \leq K|u_m|(1 + |q|)$$
$$|\frac{\partial G_t^m}{\partial u_k}| \leq K\frac{u_m}{u_k}(1 + |q|)$$

and for any simple strategy Q *the indirect utilities of the market makers solve the following stochastic differential equation:*

$$dU_t = G_t(U_t, Q_t)dW_t, \quad U_0^m = \mathbb{E}[u_m(\alpha_0^m)].$$

Stability of SDE

The construction of general strategies follows from the following

Theorem

Assume the technical conditions above. Let (Q^n) *be a sequence of simple processes and* Q *be a (general) stochastic process such that*

$$\int_0^T (Q_t^n - Q_t)^2 dt \xrightarrow{\mathbb{P}} 0, \quad n \to \infty.$$

Then the terminal values $V_T(Q^n)$ *converge in probability to*

$$V_T(Q) = \sum_{m=1}^{M} \alpha_0^m - \sum_{m=1}^{M} u_m^{-1}(U_T^m)$$

where $U = U(Q)$ *solves the following SDE:*

$$dU_t = G_t(U_t, Q_t)dW_t, \quad U_0^m = \mathbb{E}[u_m(\alpha_0^m)].$$

Remark on admissibility

The previous theorem allows us to define terminal wealth for any process Q satisfying:

$$\int_0^T Q_t^2 dt < \infty \quad (\mathbb{P} - a.s.).$$

Contrary to classical "small" agent model this set of strategies does not allow arbitrage. Indeed,

$U(Q)$ is a *local martingale bounded above* $\Rightarrow$ submartingale .

It follows that

$$\mathbb{E}[u_m(A_T^m(Q))] \geq \mathbb{E}[u_m(\alpha_0^m)], \quad 1 \leq m \leq M.$$

Hence,

$$V_T(Q) = \sum_{m=1}^{M} \alpha_0^m - \sum_{m=1}^{M} A_T^m(Q) \geq 0 \quad \Rightarrow V_T(Q) = 0.$$

Asymptotic analysis: summary of results

- For a strategy Q we have the following expansion for terminal wealth:

$$V_T(\epsilon Q) = \epsilon \int_0^T Q_u dS_u^0 + \frac{1}{2}\epsilon^2 L_T(Q),$$

  where $L_T(Q)$ can be computed by solving two auxiliary *linear* SDEs.
- We use above expansion to compute replication strategy and liquidity correction to the prices of derivatives in the next order (ϵ^2). (Good qualitative properties!)
- Liquidity correction to the prices of derivatives can also be computed using an expansion of market indifference prices. (Easier to do than hedging!).
- Key inputs: *risk-tolerance wealth processes* of market makers for initial Pareto equilibrium.

Conclusion

- We have developed a continuous-time model for large trader starting with economic primitives, namely, the preferences of market makers.
- In this model, the large investor trades "smartly", not revealing herself to market makers and, hence, not increasing their expected utilities.
- We show that the computation of terminal wealth $V_T(Q)$ for a strategy Q comes through a solution of a non-linear SDE.
- The model allows us to compute rather explicitly liquidity corrections to the terminal capitals of trading strategies and to the prices of derivatives.
- The model has "good" qualitative properties.