STOCHASTIC PORTFOLIO THEORY

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SYNOPSIS

The purpose of this talk is to offer an overview of **Stochastic Portfolio Theory**, a rich and flexible framework for analyzing portfolio behavior and equity market structure. This theory is descriptive as opposed to normative, is consistent with observable characteristics of actual markets and portfolios, and provides a theoretical tool which is useful for practical applications.

As a theoretical tool, this framework provides fresh insights into questions of market structure and arbitrage, and can be used to construct portfolios with controlled behavior. As a practical tool, Stochastic Portfolio Theory has been applied to the analysis and optimization of portfolio performance and has been the basis of successful investment strategies for close to 20 years.

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1. THE MODEL. Standard Model (Bachelier, Samuelson,...) for a financial market with n stocks and $d \geq n$ factors: for $i = 1, \ldots, n$,

$$\underbrace{dX_i(t) = X_i(t) \left[b_i(t)dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t) dW_\nu(t) \right]}\,.$$

Vector of rates-of-return: $b(\cdot) = (b_1(\cdot), \ldots, b_n(\cdot))'$. *Matrix* of volatilities: $\sigma(\cdot) = (\sigma_{i\nu}(\cdot))_{1 \leq i \leq n,\, 1 \leq \nu \leq d}$ will be assumed bounded for simplicity.

♣ Assumption: for every $T \in (0, \infty)$ we have

$$\sum_{i=1}^{n} \int_0^T \left| b_i(t) \right|^2 dt < \infty \,, \qquad \text{a.s.}$$

All processes are adapted to a given flow of information (or "filtration") $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$, which satisfies the usual conditions and may be strictly larger than the filtration generated by the driving Brownian motion $W(\cdot) = (W_1(\cdot), \ldots, W_d(\cdot))'$.

No Markovian, or Gaussian, assumptions...

Suppose, for simplicity, that the variance/covariance matrix $a(\cdot) = \sigma(\cdot)\sigma'(\cdot)$ has all its eigenvalues bounded away from zero and infinity: that is,

$$\kappa||\,\xi||^2 \le \xi' a(t)\xi \le K||\,\xi||^2, \quad \forall\ \xi \in \mathbb{R}^d$$

holds a.s. (for suitable constants $0 < \kappa < K < \infty$).

- Solution of the equation for stock-price $X_i(\cdot)$:

$$\underbrace{d\,(\log X_i(t)) \;=\; \gamma_i(t)\,dt + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\,dW_\nu(t)}$$

with

$$\boxed{\gamma_i(t) := b_i(t) - \frac{1}{2}a_{ii}(t)}$$

the *growth-rate* of the i^{th} stock, in the sense

$$\lim_{t\to\infty} \frac{1}{T}\left(\log X_i(t) - \int_0^T \gamma_i(t)dt\right) = 0 \quad \text{a.s.}$$

2. PORTFOLIO. An adapted vector process

$$\pi(t) = (\pi_1(t), \cdots, \pi_n(t))' \; ;$$

fully-invested, no short-sales, no risk-free asset:

$$\pi_i(t) \geq 0\,, \quad \sum_{i=1}^{n} \pi_i(t) = 1 \qquad \text{for all} \quad t \geq 0\,.$$

Value $Z^\pi(\cdot)$ of portfolio:

$$\frac{dZ^\pi(t)}{Z^\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dX_i(t)}{X_i(t)} = b^\pi(t)dt + \sum_{\nu=1}^{d} \sigma_\nu^\pi(t) dW_\nu(t)$$

with $Z^\pi(0) = 1$. Here

$$b^\pi(t) := \sum_{i=1}^{n} \pi_i(t) b_i(t)\,, \qquad \sigma_\nu^\pi(t) := \sum_{i=1}^{n} \pi_i(t) \sigma_{i\nu}(t)\,,$$

are, respectively, the portfolio rate-of-return, and the portfolio volatilities.

¶ Solution of this equation:

$$\underbrace{d\Big(\log Z^\pi(t) \Big) \;=\; \gamma^\pi(t)\, dt + \sum_{\nu=1}^{n} \sigma^\pi_\nu(t)\, dW_\nu(t)}\,.$$

Portfolio growth-rate

$$\gamma^\pi(t) := \sum_{i=1}^{n} \pi_i(t)\gamma_i(t) + \gamma^\pi_*(t)\,.$$

Excess growth-rate

$$\underbrace{\gamma^\pi_*(t) := \frac{1}{2}\left(\sum_{i=1}^{n} \pi_i(t) a_{ii}(t) - \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t) \right)}\,.$$

Portfolio variance

$$a^{\pi\pi}(t) := \sum_{\nu=1}^{d} (\sigma^\pi_\nu(t))^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t)\,.$$

♠ For a given portfolio $\pi(\cdot)$, let us introduce the "order statistics" notation, in decreasing order:

$$\pi_{(1)} := \max_{1 \le i \le n} \pi_i \ge \pi_{(2)} \ge \ldots \ge \pi_{(n)} := \min_{1 \le i \le n} \pi_i\,.$$

3. MARKET PORTFOLIO: Look at $X_i(t)$ as the capitalization of company i at time t. Then $X(t) := X_1(t)+\ldots+X_n(t)$ is the total capitalization of all stocks in the market, and

$$\boxed{\mu_i(t) := \frac{X_i(t)}{X(t)} = \frac{X_i(t)}{X_1(t) + \ldots + X_n(t)} > 0}$$

the *relative capitalization* of the i^{th} company. Clearly $\sum_{i=1}^{n} \mu_i(t) = 1$ for all $t \geq 0$, so $\mu(\cdot)$ is a portfolio process, called **"market portfolio"**.

. Ownership of $\mu(\cdot)$ is tantamount to ownership of the entire market, since $Z^{\mu}(\cdot) \equiv \text{c.}X(\cdot)$.

♣ **FACT 1:** $$\boxed{\gamma_*^{\pi}(\cdot) \geq (\kappa/2) \cdot \Big(1 - \pi_{(1)}(\cdot)\Big)} \,.$$

Moral: Excess-rate-of growth is non-negative, strictly positive unless the portfolio concentrates on a single stock: diversification helps not only to reduce variance, but also to “enhance growth”.

HOW? Consider, for instance, a *fixed-proportion* portfolio $\pi_i(\cdot) \equiv \mathfrak{p}_i \geq 0$ with $\sum_{i=1}^n \mathfrak{p}_i = 1$ and $\mathfrak{p}_{(1)} = 1 - \eta < 1$. Then

$$\log\left(\frac{Z^{\mathfrak{p}}(T)}{Z^{\mu}(T)}\right) - \sum_{i=1}^{n} \mathfrak{p}_i \log \mu_i(T) = \int_0^T \gamma_*^{\mathfrak{p}}(t)\, dt \geq \frac{\kappa\eta}{2} T\,.$$

And if $\lim_{T\to\infty} \frac{1}{T} \log \mu_i(T) = 0$ (no individual stock collapses very fast), then this gives almost surely

$$\underline{\lim}_{T\to\infty} \frac{1}{T} \log\left(\frac{Z^{\mathfrak{p}}(T)}{Z^{\mu}(T)}\right) \geq \frac{\kappa\eta}{2} > 0 \,:$$

a significant outperforming of the market.

`Remark:` Tom Cover's "universal portfolio"

$$\Pi_i(t) := \frac{\int_{\Delta^n} \mathfrak{p}_i\, Z^{\mathfrak{p}}(t)\, d\mathfrak{p}}{\int_{\Delta^n} Z^{\mathfrak{p}}(t)\, d\mathfrak{p}}\,, \quad i = 1, \cdots, n$$

has value

$$Z^{\Pi}(t) = \frac{\int_{\Delta^n} Z^{\mathfrak{p}}(t)\, d\mathfrak{p}}{\int_{\Delta^n} d\mathfrak{p}} \sim \max_{\mathfrak{p}\in\Delta^n} Z^{\mathfrak{p}}(t)\,.$$

♣ `FACT 2:` $$\boxed{\gamma_*^{\pi}(\cdot) \leq 2K \cdot \left(1 - \pi_{(1)}(\cdot)\right)}\,.$$

4. DIVERSITY. The market-model $\mathcal{M}$ is called

• `Diverse` on $[0,T]$, if there exists $\delta \in (0,1)$ such that we have a.s.: $\underbrace{\mu_{(1)}(t) < 1-\delta}$, $\forall\ 0 \le t \le T$.

• `Weakly Diverse` on $[0,T]$, if for some $\delta \in (0,1)$:

$$\underbrace{\frac{1}{T}\int_0^T \mu_{(1)}(t)\,dt \,<\, 1-\delta\,, \quad \text{a.s.}}$$

♣ `FACT 3:` *If $\mathcal{M}$ is diverse, then $\gamma_*^{\mu}(\cdot) \ge \zeta$ for some $\zeta > 0$; and vice-versa.*

♣ `FACT 4:` *If all stocks $i = 1, \ldots, n$ in the market have the same growth-rate $\gamma_i(\cdot) \equiv \gamma(\cdot)$, then*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \gamma_*^{\mu}(t)\,dt \,=\, 0, \quad \textit{a.s.}$$

In particular, such a market **cannot be diverse** *on long time-horizons*: once in a while a single stock dominates such a market, then recedes; sooner or later another stock takes its place as absolutely dominant leader; and so on.

• Here is a quick argument: from $\gamma_i(\cdot) \equiv \gamma(\cdot)$ and $X(\cdot) = X_1(\cdot) + \cdots + X_n(\cdot)$ we have

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X(T) - \int_0^T \gamma^\mu(t)dt\right) = 0\,,$$

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma(t)dt\right) = 0\,.$$

for all $1 \le i \le n$. But then

$$\lim_{T\to\infty} \frac{1}{T}\left(\log X_{(1)}(T) - \int_0^T \gamma(t)dt\right) = 0\,, \quad \text{a.s.}$$

for the biggest stock $X_{(1)}(\cdot) := \max_{1\le i\le n} X_i(\cdot)$, and note $X_{(1)}(\cdot) \le X(\cdot) \le n\,X_{(1)}(\cdot)$. Therefore,

$$\lim_{T\to\infty} \frac{1}{T}\Big(\log X_{(1)}(T) - \log X(T) \Big) = 0\,, \quad \text{thus}$$

$$\lim \frac{1}{T}\int_0^T \Big(\gamma^\mu(t) - \gamma(t)\Big)\,dt \,=\, 0\,.$$

But $\gamma^\mu(t) = \sum_{i=1}^n \mu_i(t)\gamma(t) + \gamma_*^\mu(t) = \gamma(t) + \gamma_*^\mu(t)$, because all growth rates are equal. $\square$

♣ `FACT 5:` *In a Weakly Diverse Market there exist potfolios* $\pi(\cdot)$ *that lead to* **arbitrage** *relative to the market-portfolio: with common initial capital* $Z^{\pi}(0) = Z^{\mu}(0) = 1$ *, and some* $T \in (0, \infty)$*, we have*

$$\mathbb{P}[Z^{\pi}(T) \geq Z^{\mu}(T)] = 1\,, \qquad \mathbb{P}[Z^{\pi}(T) > Z^{\mu}(T)] > 0\,.$$

And not only do such relative arbitrages exist; they can be described, even constructed, fairly explicitly.

5. DIVERSITY-WEIGHTING & ARBITRAGE

For fixed $p \in (0, 1)$, set

$$\boxed{\pi_i(t) \equiv \pi_i^{(p)}(t) := \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}\,, \quad i = 1, \ldots, n}\,.$$

Relative to the market portfolio $\mu(\cdot)$, this $\pi(\cdot)$ decreases slightly the weights of the largest stock(s), and increases slightly those of the smallest stock(s), while *preserving the relative rankings of all stocks*.

We shall show that, in a weakly-diverse market $\mathcal{M}$, this portfolio

$$\underbrace{\pi_i(t) \equiv \pi_i^{(p)}(t) := \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}}, \quad i = 1, \ldots, n,$$

for some fixed $p \in (0,1)$, satisfies

$$\boxed{\mathbb{P}[Z^\pi(T) > Z^\mu(T)] = 1, \quad \forall \;\; T \geq T_* := \frac{2}{p\kappa\delta} \cdot \log n}\;.$$

In particular, $\pi^{(p)}(\cdot)$ represents an arbitrage opportunity relative to the market-portfolio $\mu(\cdot)$.

✠ Suitable modifications of $\pi^{(p)}(\cdot)$ can generate such arbitrage over <u>*arbitrary*</u> time-horizons.

The significance of such a result, for practical long-term portfolio management, cannot be overstated. Discussion and performance charts can be found in the monograph Fernholz (2002).

6: Performance of Diversity-Weighting

Indeed, for this *"diversity-weighted"* portfolio

$$\pi_i(t) \equiv \pi_i^{(p)}(t) := \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p}, \qquad i = 1, \ldots, n$$

with fixed $0 < p < 1$ and $\mathbf{D}(x) := \left(\sum_{j=1}^n x_j^p \right)^{1/p}$, we have

$$\boxed{\log \left(\frac{Z^\pi(T)}{Z^\mu(T)} \right) = \log \left(\frac{\mathbf{D}(\mu(T))}{\mathbf{D}(\mu(0))} \right) + (1-p) \int_0^T \gamma_*^\pi(t) dt}.$$

- First term on RHS tends to be mean-reverting, and is certainly bounded:

$$1 = \sum_{j=1}^n x_j \le \sum_{j=1}^n (x_j)^p \le \Big(\mathbf{D}(x) \Big)^p \le n^{1-p}.$$

Measure of Diversity: minimum occurs when one company is the entire market, maximum when all companies have equal relative weights.

- We remarked already, that the biggest weight of $\pi(\cdot)$ does not exceed the largest market weight:

$$\pi_{(1)}(t) := \max_{1 \le i \le n} \pi_i(t) = \frac{\big(\mu_{(1)}(t)\big)^p}{\sum_{k=1}^n \big(\mu_{(k)}(t)\big)^p} \le \mu_{(1)}(t)\,.$$

By weak diversity over $[0,T]$, there is a number $\delta \in (0,1)$ for which $\int_0^T (1-\mu_{(1)}(t))\,dt > \delta\, T$ holds; thus, from Fact # 1:

$$\frac{2}{\kappa} \cdot \int_0^T \gamma_*^\pi(t)\,dt \ \ge\ \int_0^T \Big(1 - \pi_{(1)}(t)\Big)\,dt$$

$$\ge \int_0^T \Big(1 - \mu_{(1)}(t)\Big)\,dt \ >\ \delta\, T\,, \quad \text{a.s.}$$

- From these two observations we get

$$\log\left(\frac{Z^\pi(T)}{Z^\mu(T)}\right) \ >\ (1-p)\left[\frac{\kappa T}{2} \cdot \delta \ -\ \frac{1}{p} \cdot \log n\right],$$

so for a time-horizon

$$T > T_* := (2 \log n)/p\kappa\delta$$

sufficiently large, the RHS is strictly positive. □

`Remark:` It can be shown similarly that, over sufficiently long time-horizons $T > 0$, arbitrage relative to the market can be constructed under

$$\boxed{\gamma^*_\mu(t) \geq \zeta > 0\,, \quad 0 \leq t \leq T} \tag{1}$$

for some real ζ, or even under the weaker condition $\int_0^T \gamma^*_\mu(t) \geq \zeta\, T > 0$.

Open Question, *whether this can also be done over arbitrary horizons* $T > 0$.

- This result does not presuppose any condition on the covariance structure $(a_{ij}(\cdot))$ of the market, beyond (1). There are examples, such as the *volatility-stabilized model* (with $\alpha \geq 0$)

$$d \log X_i(t) = \frac{\alpha}{2\mu_i(t)}\, dt + \frac{dW_i(t)}{\sqrt{\mu_i(t)}}\,, \quad i = 1, \cdots, n$$

for which variances are unbounded, diversity fails, but (1) holds: $\gamma^*_\mu(\cdot) \equiv ((1+\alpha)n-1)/2$, $a^{\mu\mu}(\cdot) \equiv 1$.

In this example, arbitrage relatively to the market can be constructed over *arbitrary* time-horizons.

7: STRICT SUPERMARTINGALES.

The existence of relative arbitrage precludes the existence of an equivalent martingale measure (EMM) – at least when the filtration $\mathbb{F}$ is generated by the Brownian motion W itself, as we now assume.

♠ In particular, if we can find a "market-price-of-risk" process $\vartheta(\cdot)$ with

$$\boxed{\sigma(\cdot)\,\vartheta(\cdot) \,=\, b(\cdot)} \quad \text{and} \quad \boxed{\int_0^T ||\vartheta(t)||^2\,dt \,<\, \infty \quad \text{a.s.}}\,,$$

then it can be shown that the exponential process

$$\underbrace{L(t) \,:=\, \exp\left\{ -\int_0^t \vartheta'(s)\,dW(s) - \frac{1}{2}\int_0^t ||\vartheta(s)||^2\,ds \right\}}$$

is a local (and super-)martingale, but *not* a martingale: $\mathbb{E}[L(T)] < 1$.

Same for $L(\cdot)X_i(\cdot)$: $\mathbb{E}[L(T)X_i(T)] < X_i(0)$.
Typically: $\lim_{T\to\infty}\mathbb{E}[L(T)X_i(T)] = 0\,,\ i = 1,\cdots,n.$

Examples of `diverse` and `volatility-stabilized` markets satisfying these conditions can be constructed.

In terms of this exponential supermartingale $L(\cdot)$, we can answer some basic questions, for $d=n$:

Q.1: `On a given time-horizon` $[0,T]$, `what is the maximal relative return in excess of the market`

$$\mathbf{R}(T) := \sup\{\, r>1 : \exists\, h(\cdot) \text{ s.t. } Z^h(T)/Z^\mu(T) \ge r\,, \text{a.s.}\,\}$$

`that can be attained by trading strategies` $h(\cdot)$?

(These can sell stock short, or invest/borrow in a money market at rate $r(\cdot)$, but are required to remain solvent: $Z^h(t) \ge 0\,,\ \forall\, 0 \le t \le T\,$.)

Q.2: `Again using such strategies, what is the shortest amount of time required to guarantee a return of at least` $r>1$, `times the market?`

$$\mathbf{T}(r) := \inf\{\, T>0 : \exists\, h(\cdot) \text{ s.t. } Z^h(T)/Z^\mu(T) \ge r\,, \text{a.s.}\,\}$$

¶ *Answers:* $\boxed{\mathbf{R}(T) = 1/f(T)}$ and $\boxed{f\Big(\mathbf{T}(r)\Big) = r}$,

$$\text{where } f(t) := \mathbb{E}\left[e^{-\int_0^t r(s)ds}\, L(t) \cdot \frac{X(t)}{X(0)} \right] \downarrow 0\,.$$

8. COMPLETENESS AND OPTIMIZATION WITHOUT EMM

In a similar vein, given an $\mathcal{F}(T)-$measurable random variable $Y : \Omega \to [0, \infty)$ (contingent claim), we can ask about its "hedging price"

$$\underbrace{\mathcal{H}^Y(T) := \inf\{\, w > 0 : \exists\, h(\cdot) \text{ s.t. } Z^{w,h}(T) \geq Y \,, \text{a.s.} \,\}}\,,$$

the smallest amount of initial capital needed to hedge it without risk.

With $D(T) := e^{-\int_0^T r(s)ds}$, this can be computed as

$$\boxed{\mathcal{H}^Y(T) = \mathsf{y} := \mathbb{E}\,[\,L(T) D(T)\, Y\,]}$$

(extended Black-Scholes) and an optimal strategy $\widehat{h}(\cdot)$ is identified via $\underline{Z^{\,\mathsf{y},\widehat{h}}(T) = Y}$, a.s.

• To wit: such a market is **complete**, despite the fact that no EMM exists for it.

♠ Take $Y = (X_1(T) - q)^+$ as an example, and assume $r(\cdot) \geq r > 0$. Simple computation $\oplus$ Jensen:

$$X_1(0) > \mathcal{H}^Y(T) \geq \Big(\mathbb{E}[L(T)D(T)X_1(T)] - qe^{-rT}\Big)^+ .$$

Letting $T \to \infty$ we get, as we have seen:

$$\mathcal{H}^Y(\infty) := \lim_{T\to\infty} \mathcal{H}^Y(T)$$

$$= \lim_{T\to\infty} \mathbb{E}[L(T)D(T)X_1(T)] = 0 .$$

. Please contrast this, to the situation whereby an EMM exists on every finite time-horizon $[0, T]$. Then at $t = 0$, you have to pay full stock-price for an option that you can never exercise!

$$\boxed{\mathcal{H}^Y(\infty) = X_1(0) .}$$

Moral: In some situations, particularly on "long" time-horizons, it might not be such a great idea to postulate the existence of EMM's.

♣ Ditto with **portfolio optimization.** Suppose we are given initial capital $w > 0$, finite time-horizon $T > 0$, and *utility function* $u : (0, \infty) \to \mathbb{R}$ (strictly increasing and concave, of class $\mathcal{C}^1$, with $u'(0+) = \infty$, $u'(\infty) = 0$.) Compute the maximal expected utility from terminal wealth

$$\mathfrak{U}(w) := \sup_{h(\cdot)} \mathbb{E}\Big[u\Big(Z^{w,h}(T) \Big) \Big],$$

decide whether the supremum is attained and, if so, identify an optimal trading strategy $\widehat{h}(\cdot)$.

. *Answer:* replicating trading strategy $\widehat{h}(\cdot)$ for the contingent claim

$$\boxed{\mathsf{Y} = I\Big(\Xi(w)\, D(T)L(T) \Big)}, \quad \text{i.e.,} \quad \boxed{Z^{w,\widehat{h}}(T) = \mathsf{Y}\,.}$$

Here $I(\cdot)$ is the inverse of the strictly decreasing "marginal utility" function $u'(\cdot)$, and $\Xi(\cdot)$ the inverse of the strictly decreasing function

$$\mathcal{W}(\xi) := \mathbb{E}\Big[D(T)L(T)\, I\left(\xi\, D(T)L(T)\right) \Big], \quad \xi > 0\,.$$

No assumption at all that $L(\cdot)$ should be a martingale, or that an EMM should exist !

✠ Here are some other problems, in which no EMM assumption is necessary:

#1: Quadratic criterion, linear constraint (Markowitz, 1952). *Minimize the portfolio variance*

$$a^{\pi\pi}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t)$$

among all portfolios $\pi(\cdot)$ *with rate-of-return*

$$b^{\pi}(t) = \sum_{i=1}^{n} \pi_i(t) b_i(t) \geq b_0$$

at least equal to a given constant.

#2: Quadratic criterion, quadratic constraint. *Minimize the portfolio variance*

$$a^{\pi\pi}(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t)$$

among all portfolios $\pi(\cdot)$ *with growth-rate at least equal to a given constant* γ_0:

$$\sum_{i=1}^{n} \pi_i(t) b_i(t) \geq \gamma_0 + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t)\,.$$

#3: *Maximize the probability of reaching a given "ceiling" $\mathfrak{c}$ before reaching a given "floor" $\mathfrak{f}$, with* $0 < \mathfrak{f} < 1 < \mathfrak{c} < \infty$. More specifically, maximize $\mathbb{P}[\mathfrak{T}_\mathfrak{c} < \mathfrak{T}_\mathfrak{f}]$, with $\mathfrak{T}_c := \inf\{t \geq 0 : Z^\pi(t) = c\}$.

In the case of constant coëfficients γ_i and a_{ij}, the solution to this problem is find a portfolio π that maximizes the mean-variance, or *signal-to-noise*, ratio (Pestien & Sudderth, MOR 1985):

$$\frac{\gamma^\pi}{a^{\pi\pi}} = \frac{\sum_{i=1}^n \pi_i(\gamma_i + \frac{1}{2}a_{ii})}{\sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} \pi_j} - \frac{1}{2},$$

#4: *Minimize the expected time* $\mathbb{E}[\mathfrak{T}_\mathfrak{c}]$ *until a given "ceiling"* $\mathfrak{c} \in (1, \infty)$ *is reached.*

Again with constant coëfficients, it turns out that it is enough to maximize the drift in the equation for $\log Z^{w,\pi}(\cdot)$, namely

$$\gamma^\pi = \sum_{i=1}^n \pi_i \left(\gamma_i + \frac{1}{2}a_{ii}\right) - \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \pi_i a_{ij} \pi_j,$$

the portfolio growth-rate (Heath, Orey, Pestien & Sudderth, SICON 1987).

#5: *Maximize the probability* $\mathbb{P}\,[\,\mathfrak{T}_{\mathfrak{c}} < T \wedge \mathfrak{T}_{\mathfrak{f}}\,]$ *of reaching a given "ceiling"* $\mathfrak{c}$ *before reaching a given "floor"* $\mathfrak{f}$ *with* $0 < \mathfrak{f} < 1 < \mathfrak{c} < \infty$*, by a given "deadline"* $T \in (0, \infty)$*.*

Always with constant coëfficients, suppose there is a portfolio $\widehat{\pi} = (\widehat{\pi}_1, \ldots, \widehat{\pi}_n)'$ that maximizes *both* the signal-to-noise ratio *and* the variance,

$$\frac{\gamma^{\pi}}{a^{\pi\pi}} = \frac{\sum_{i=1}^{n} \pi_i (\gamma_i + \frac{1}{2} a_{ii})}{\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_i a_{ij} \pi_j} - \frac{1}{2} \qquad \text{and} \qquad a^{\pi\pi}\,,$$

over all $\pi_1 \geq 0, \ldots, \pi_n \geq 0$ with $\sum_{i=1}^{n} \pi_i = 1$. Then this portfolio $\widehat{\pi}$ is optimal for the above criterion (Sudderth & Weerasinghe, MOR 1989).

This is a big assumption; it is satisfied, for instance, under the (very stringent) condition that, for some $G > 0$, we have

$$b_i = \gamma_i + \frac{1}{2} a_{ii} = -G\,, \quad \text{for all } i = 1, \ldots, n\,.$$

`Open Question:` As far as I can tell, nobody seems to know the solution to this problem, if such "simultaneous maximization" is not possible.

9. SOME CONCLUDING REMARKS

We have surveyed a framework, called *Stochastic Portfolio Theory*, for studying the behavior of portfolio rules – and exhibited simple conditions, such as "diversity" (there are others...), which can lead to arbitrages relative to the market.

All these conditions, diversity included, are **descriptive** as opposed to normative, and can be tested from the predictable characteristics of the model posited for the market. In contrast, familiar assumptions, such as the existence of an equivalent martingale measure (EMM), are **normative** in nature, and *cannot* be decided on the basis of predictable characteristics in the model; see example in [KK] (2006).

The existence of such relative arbitrage is not the end of the world; it is not heresy, or scandal, either.

Under reasonably general conditions, one can still work with appropriate "deflators" $L(\cdot)D(\cdot)$ for the purposes of hedging derivatives and of portfolio optimization.

Considerable computational tractability is lost, as the marvelous tool that is the EMM goes out of the window; nevertheless, big swaths of the field of Mathematical Finance remain totally or mostly intact, and completely new areas and issues thrust themselves onto the scene.

There is a lot more scope to this *Stochastic Portfolio Theory* than can be covered in one talk. For those interested, there is the survey paper with R. Fernholz, at the bottom of the page

www.math.columbia.edu/ $\sim$ ik/preprints.html

It contains a host of open problems.

Please let us know if you solve some of them!