Dynamical modelling of successive defaults

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Outline

Motivation and Introduction

- Rapid development of credit portfolio products : k^{th}-to-default swap, CDOs
- A practical question proposed by the practitioners: possibility of a recursive procedure to study the successive defaults?
- Calculation of conditional expectations $\mathbb{E}[Y_T|\mathcal{G}_t]$ when $(\mathcal{G}_t)_{t\geq 0}$ is some large filtration including default informations
- Study of the "after-default case" by using the density process approach

An illustrative example

An illustrative example

- A simple deterministic model of two credits, denote by $\tau = \min(\tau_1, \tau_2)$.
- Observable information : whether the first default occurs.
- The basic hypothesis is based on a stationary point of view of the practitioners

$$\mathbb{P}(\tau_i > T \mid \tau > t) = e^{-\mu^i(t)\cdot(T-t)}, \quad (i = 1, 2)$$

where $\mu^i(t)$ characterizes the individual default and it can be renewed with market information at t.

- The marginal distributions remain in the exponential family.

The joint probability

- Let $\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) = \mathbb{P}(\tau_1 > t_1)\mathbb{P}(\tau_2 > t_2)\rho(t_1, t_2)$. Consider the survival copula function $\widetilde{C}(u, v)$ such that $\widetilde{C}(\mathbb{P}(\tau_1 > t_1), \mathbb{P}(\tau_2 > t_2)) = \mathbb{P}(\tau_1 > t_1, \tau_2 > t_2)$, then

$$\widetilde{C}(u,v) = \begin{cases} uv\rho\left(\frac{\ln u}{\mu^1(0)}, \frac{\ln v}{\mu^2(0)}\right), & \text{if} \quad u, v > 0; \\ 0, & \text{if} \quad u = 0 \quad \text{or} \quad v = 0. \end{cases}$$

- Joint probability : If $\rho(t_1, t_2) \in C^{1,1}$, then the joint survival probability is given by

$$\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) = \exp\left(- \int_0^{t_1} \mu^1(s \wedge t_2)ds - \int_0^{t_2} \mu^2(s \wedge t_1)ds \right).$$

- First observation: The joint probability function depends on all the dynamics of the marginal distributions and the copula can not be chosen independently with marginal distributions.

The joint probability

- **Proposition:** If $\rho(t_1, t_2) \in C^2$ and if $\mu^1(t), \mu^2(t) \in C^1$, then

$$\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2) = \exp\Big(- \mu^1(0)t_1 - \mu^2(0)t_2 + \int_0^{t_1 \wedge t_2} \varphi(s)(t_1 + t_2 - 2s)ds \Big).$$

where $\varphi(t) = \frac{\partial^2}{\partial t_1 \partial t_2}\Big|_{t_1 = t_2 = t} \ln \rho(t_1, t_2)$. In addition, we have

$$\mu^i(t) = \mu^i(0) - \int_0^t \varphi(s)ds.$$

- μ^1 and μ^2 follow the same dynamics apart from their initial values due to the symmetric information flow and the stationary property.

First default and contagious jumps

- The distribution of the first default is given by $\mathbb{P}(\tau > t) = \exp\Big(- \int_0^t \mu^1(s) + \mu^2(s)ds\Big)$;
- For the surviving credit, it becomes complicated. Let $\mathcal{D}_t = \mathcal{D}_t^1 \vee \mathcal{D}_t^2$ where $\mathcal{D}_t^i = \sigma(1\!\!1_{\{\tau_i \le s\}}, s \le t)$ $(i = 1, 2)$. Then

$$\mathbb{E}[1\!\!1_{\{\tau_i > T\}} | \mathcal{D}_t] = 1\!\!1_{\{\tau > t\}} \exp\Big(-(\mu^i(0) - \int_0^t \varphi(s)ds)(T - t)\Big)$$
$$+ 1\!\!1_{\{\tau_i > t, \tau_j \le t\}} \exp\Big(-(\mu^i(0) - \int_0^\tau \varphi(s)ds)(T - t)\Big)$$
$$\cdot \frac{\mu^j(0) - \varphi(\tau)(T - \tau) - \int_0^\tau \varphi(s)ds}{\mu^j(0) - \varphi(\tau)(t - \tau) - \int_0^\tau \varphi(s)ds}.$$

- We observe the contagious jump phenomenon

The second default time

- Second observation on $\sigma = \max(\tau_1, \tau_2)$: the conditional distribution $\mathbb{E}[1\!\!1_{\{\sigma > T\}}|\mathcal{D}_t^\tau]$ can not remain in the exponential family neither on $\{\tau > t\}$ nor on $\{\tau \leq t\}$, except when τ_1 and τ_2 are independent and identically distributed.
- Remark: conditioned on the first default, these exists no longer the stationary property, as expected by some market practitioners!
- We need to study the successive defaults in an abstract way.

The General Case

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the density process framework

Before-default case and Minimal assumption

- Let $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ be a filtered probability space representing the market. The filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ represents the global information on the market.
- Let τ be a finite $\mathbb{G}$-stopping time.
- Consider a subfiltration $\mathbb{F}$ of $\mathbb{G}$ satisfying the following condition presented by Jeulin and Yor (1978), Jacod (1982).
- (Minimal Assumption): We say $(\mathbb{F}, \mathbb{G}, \tau)$ satisfy the Minimal Assumption (MA) if for any $t \geq 0$ and any $U \in \mathcal{G}_t$, there exists $V \in \mathcal{F}_t$ such that $U \cap \{\tau > t\} = V \cap \{\tau > t\}$.

Before-default case and Minimal assumption

Two examples of $(\mathbb{F}, \mathbb{G}, \tau)$ satisfing MA:

- In the single credit case, τ represents one default time and let $\mathbb{D} = (\mathcal{D}_t)_{t \geq 0}$ where $\mathcal{D}_t = \sigma(1\!\!1_{\{\tau \leq s\}}, s \leq t)$. $\mathbb{F}$ satisfies $\mathbb{G} = \mathbb{D} \vee \mathbb{F}$.
- In the multi-credits case, τ represents the first default time $\tau = \min(\tau_1, \cdots, \tau_n)$ and $\mathbb{F}$ satisfies $\mathbb{G} = \mathbb{F} \vee \mathbb{D}^1 \cdots \vee \mathbb{D}^n$.

A direct consequence: Assume that $(\mathbb{F}, \mathbb{G}, \tau)$ satisfy MA. For any $\mathcal{G}$-measurable random variable Y, if $\mathbb{P}(\tau > t | \mathcal{F}_t) > 0$, a.s. then

$$\mathbb{E}[1\!\!1_{\{\tau > t\}} Y | \mathcal{G}_t] = 1\!\!1_{\{\tau > t\}} \frac{\mathbb{E}[1\!\!1_{\{\tau > t\}} Y | \mathcal{F}_t]}{\mathbb{E}[1\!\!1_{\{\tau > t\}} | \mathcal{F}_t]}.$$

After-default case and Density process

(H_J Hypothesis, Jacod (82)) For any $t, \theta \geq 0$, we assume that there exists a family of $\mathbb{F}$-adapted processes, called the density process $(\alpha_t(u), t \geq 0)$, such that

$$S_t(\theta) = \mathbb{P}(\tau > \theta | \mathcal{F}_t) = \int_\theta^\infty \alpha_t(u) du.$$

Proposition

For any $t, u \geq 0$, let $Y(t,u)$ be a random variable such that $Y(t,u) \in \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R})$. If $\mathbb{G} = \mathbb{F} \vee \mathbb{D}$ and if $\alpha_t(u) > 0$, then for any $0 \leq t \leq T$,

$$E[Y(T,\tau)|\mathcal{G}_t] 1\!\!1_{\{\tau \leq t\}} = \frac{E\left[Y(T,s)\alpha_T(s) \middle| \mathcal{F}_t\right]}{\alpha_t(s)}\Big|_{s=\tau} 1\!\!1_{\{\tau \leq t\}}.$$

Density and intensity processes

- The $\mathbb{G}$-compensator process Λ of τ is a predictable process such that $(N_t = 1\!\!1_{\{\tau \leq t\}} - \Lambda_t, t \geq 0)$ is a $\mathbb{G}$-martingale. If $\Lambda_t = \int_0^t \lambda_s^{\mathbb{G}} ds$, then $\lambda^{\mathbb{G}}$ is called the $\mathbb{G}$-intensity process.
- Proposition: Assume that $(\mathbb{F}, \mathbb{G}, \tau)$ satisfy minimal assumption. If the survival density $\alpha_t(u)$ exists, then the $\mathbb{G}$-compensator process Λ of τ is given by

$$d\Lambda_t = 1\!\!1_{]0,\tau]}(t) \frac{\alpha_t(t)}{\int_t^\infty \alpha_t(u)du} dt.$$

Density and intensity processes

- The intensity process can be deduced from the density process. However, the reverse is not true in general.
- Proposition : Assume that $(\mathbb{F}, \mathbb{G}, \tau)$ satisfy minimal assumption. If the $\mathbb{G}$-intensity process $(\lambda_t^{\mathbb{G}}, t \geq 0)$ of τ exists. Then, for any $u \geq t$, the density of the conditional survival law of τ is given by

$$\alpha_t(u) = \mathbb{E}\big[\lambda_u^{\mathbb{G}} | \mathcal{F}_t\big].$$

- The after-default case requires us to know $\alpha_t(u)$ for $u < t$, which can not be obtained from the intensity process.

Successive defaults

Two ordered default times - the first default

- Consider τ_1 and τ_2 with associated filtrations $\mathbb{D}^1$ and $\mathbb{D}^2$. Let $\mathbb{F}$ such that $\mathbb{G} = \mathbb{F} \vee \mathbb{D}^1 \vee \mathbb{D}^2$.
- Let $\tau = \min(\tau_1, \tau_2)$ and $\sigma = \max(\tau_1, \tau_2)$ with $\mathbb{D}^{(1)}$ and $\mathbb{D}^{(2)}$.
- $(\mathbb{F}, \mathbb{G}, \tau)$ satisfy the minimal assumption. Hence the first default can be treated in the same way as for a single credit.
- Proposition: Assume that the joint density process of (τ_1, τ_2) exists, i.e.

$$\mathbb{P}(\tau_1 > t_1, \tau_2 > t_2 \,|\, \mathcal{F}_t) = \int_{t_1}^{\infty} du_1 \int_{t_2}^{\infty} du_2 \, p_t(u_1, u_2).$$

Then the density process $(\alpha_t^\tau(\theta), t \geq 0)$ of τ is given by

$$\alpha_t^\tau(\theta) = \int_\theta^\infty du \left(p_t(\theta, u) + p_t(u, \theta) \right).$$

Two ordered default times - between two defaults

- The period between two defaults corresponds to before-default case of σ and after-default case of τ.
- Let $\mathbb{G}^{(1)} = \mathbb{F} \vee \mathbb{D}^{(1)}$ and $\mathbb{G}^{(2)} = \mathbb{G}^{(1)} \vee \mathbb{D}^{(2)}$. We calculate $\mathbb{G}^{(2)}$-conditional expectations by a recursive way.
- Corollary : Assume that the conditional density process of $S_t^{(2|1)}(\theta) := \mathbb{P}(\sigma > \theta | \mathcal{G}_t^{(1)}) = \int_\theta^\infty \alpha_t^{(2|1)}(u)du$ exists for all $t, \theta \geq 0$. Let $Y(T, t_1, t_2)$ be an $\mathcal{F}_T$-measurable r.v. such that $Y(., t_1, t_2)$ is a Borel function. Then

$$\mathbb{E}[Y(T,\tau,\sigma)|\mathcal{G}_t^{(2)}]1\!\!1_{\{\tau\leq t<\sigma\}} = \frac{\mathbb{E}\left[\int_t^\infty du_2 Y(T,\tau,u_2)\alpha_T^{(2|1)}(u_2)|\mathcal{G}_t^{(1)}\right]}{\int_t^\infty du_2\, \alpha_t^{(2|1)}(u_2)} 1\!\!1_{\{\tau\leq t<\sigma\}}$$

Two ordered default times - between two defaults

- Furthermore, we can bring all calculations to $\mathbb{F}$-conditional expectations.
- Proposition : Assume that the joint density process $(\alpha_t(t_1, t_2), t \geq 0)$ of (τ, σ) exists for all $t_1, t_2 \geq 0$. Then

$$\alpha_t^{(2|1)}(\theta) = 1\!\!1_{\{\tau > t\}} \frac{\int_t^\infty du_1 \alpha_t(u_1, \theta)}{\int_t^\infty du_1 \int_{u_1}^\infty du_2 \alpha_t(u_1, u_2)} + 1\!\!1_{\{\tau \leq t\}} \frac{\alpha_t(\tau, \theta)}{\int_\tau^\infty du_2 \alpha_t(\tau, u_2)}.$$

Moreover,

$$\mathbb{E}[Y(T, \tau, \sigma)|\mathcal{G}_t^{(2)}] 1\!\!1_{\{\tau \leq t < \sigma\}} = 1\!\!1_{\{\tau \leq t < \sigma\}} \mathbb{E}\left[\frac{\int_t^\infty dv Y(T, u, v)\alpha_T(u, v)}{\int_t^\infty dv \alpha_t(u, v)} \Big| \mathcal{F}_t \right] \Bigg|_{u=\tau}.$$

Intensity process of τ and σ

- Denote by Λ^i the $\mathbb{G}$-compensator process of τ_i
- For the first default : If $\mathbb{P}(\tau_i = \tau_2) = 0$, then

$$\Lambda^{\tau}_{t\wedge\tau} = \Lambda^{1}_{t\wedge\tau} + \Lambda^{2}_{t\wedge\tau}.$$

- For the second default: by the recursive method, we have

$$d\Lambda^{\sigma}_t = 1\!\!1_{[\tau,\sigma]}(t)\frac{\alpha_t(\tau,t)}{\int_t^\infty du_2\alpha_t(\tau,u_2)}\,dt$$

Joint density processes

- Calculations are determined by the $\mathbb{F}$-adapted process $(\alpha_t(t_1, t_2), t \geq 0)$.
- Similar results exist for τ_1 and τ_2 following 4 possible default scenarios, using the non-ordered denstiy process $(p_t(t_1, t_2), t \geq 0)$.
- Proposition : For any $t, t_1, t_2 \geq 0$,

$$\alpha_t(t_1, t_2) = 1\!\!1_{\{t_1 \leq t_2\}} \big(p_t(t_1, t_2) + p_t(t_2, t_1)\big).$$

- Modelling of $(p_t(t_1, t_2), t \geq 0)$!

Generalization and application

- Consider ordered $\mathbb{G}$-stopping times $\sigma_1 \leq \cdots \leq \sigma_n$,

$$\mathbb{E}\big[Y(T,\sigma_1,\cdots,\sigma_n)|\mathcal{G}_t^{(1,\cdots,n)}\big] = \sum_{i=1}^{n} 1\!\!1_{\{\sigma_i\leq t,\sigma_{i+1}>t\}} \cdot$$

$$\frac{\mathbb{E}\big[\int_{t\vee u_i}^{\infty} du_{i+1}\int_{u_{i+1}}^{\infty}\cdots\int_{u_{n-1}}^{\infty} du_n Y(T,u_1,\cdots,u_n)\,\alpha_T(u_1,\cdots,u_n)|\mathcal{F}_t\big]}{\int_{t\vee u_i}^{\infty} du_{i+1}\int_{u_{i+1}}^{\infty}\cdots\int_{u_{n-1}}^{\infty} du_n\,\alpha_t(u_1,\cdots,u_n)}\Bigg|_{\substack{u_1=\sigma_1\\ \cdots\\ u_i=\sigma_i}}$$

When $i = n$, we use convention $\sigma_{n+1} = \infty$.

Application

- Pour les CDOs, let $l_T = \sum_{i=1}^n 1\!\!1_{\{\tau_i \le T\}}$, then

$$\mathbb{E}\big[(K - l_T)^+ | \mathcal{G}_t^{(1,\cdots,n)}\big] = \int_{-\infty}^K \mathbb{E}\big[1\!\!1_{\{l_T \le K\}} | \mathcal{G}_t^{(1,\cdots,n)}\big] dK$$
$$= \int_{-\infty}^K \mathbb{E}\big[1\!\!1_{\{\sigma_{[K]+1} > T\}} | \mathcal{G}_t^{(1,\cdots,n)}\big] dK.$$

- For any $m \ge 0$,

$$\mathbb{E}\big[1\!\!1_{\{\sigma_m > T\}} | \mathcal{G}_t^{(1,\cdots,n)}\big] = \sum_{j=1}^{m-1} 1\!\!1_{\{\sigma_j \le t < \sigma_{j+1}\}}.$$
$$\frac{\mathbb{E}\big[\int_{t \vee u_j}^\infty du_{j+1} \int_{u_{j+1}}^\infty \cdots \int_{u_{n-1}}^\infty du_n 1\!\!1_{\{u_m > T\}} \alpha_T(u_1, \cdots, u_n) | \mathcal{F}_t\big]}{\int_{t \vee u_j}^\infty du_{j+1} \cdots \int_{u_{n-1}}^\infty du_n \, \alpha_t(u_1, \cdots, u_n)}\Bigg|_{\substack{u_1 = \sigma_1 \\ \cdots \\ u_j = \sigma_j}}$$