Adaptive integration for multi-factor portfolio credit loss models

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[1] Joint work with Cornelis W. Oosterlee at TU Delft and CWI.

OUTLINE

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   - Adaptive Genz-Malik Rule
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Portfolio credit loss

- A credit portfolio consisting of n obligors with exposure $w_1, w_2, \ldots, w_n$.
- Default indicator $D_i = 1_{\{X_i < \gamma_i\}}$, X_i standardized log asset value and γ_i default threshold.
- Default probability $p_i = P(X_i < \gamma_i)$.
- Portfolio loss $L = \sum_{i=1}^{n} w_i D_i$.
- Tail probability $P(L > x)$ for some extreme loss level x.
- Value at Risk (VaR): the α-quantile of the loss distribution of L for some α close to 1.

Latent factor models

$$X_i = \alpha_{i1} Y_1 + \cdots + \alpha_{id} Y_d + \beta_i Z_i = \alpha_i \mathbf{Y^d} + \beta_i Z_i,$$

- $Y_1 \ldots Y_d$: systematic factors that affect more than one obligor, e.g., state of economy, effects of different industries and geographical regions.
- Z_i: idiosyncratic factor that only affects an obligor itself.
- $\mathbf{Y^d}$ and Z_i are independent for all i.
- $D_i\left(\mathbf{Y^d}\right)$ and $D_j\left(\mathbf{Y^d}\right)$ are independent.
- $L\left(\mathbf{Y^d}\right) = \sum w_i D_i\left(\mathbf{Y^d}\right)$ becomes a weighted sum of independent Bernoulli random variables.

Tail probability: a numerical integration problem

$$P(L > x) = \int P\left(L > x \,|\, \mathbf{Y^d}\right) dP(\mathbf{Y^d})$$

The integrand $P\left(L > x \,|\, \mathbf{Y^d}\right)$ can be calculated accurately by

- the recursive method - Andersen et al (2003)
- the normal approximation - Martin (2004)
- the saddlepoint approximation - Martin et al (2001a, b), Huang et al (2007a)
- review of various methods - Glasserman and Ruiz-Mata (2006), Huang et al (2007b)

PROPERTIES OF THE CONDITIONAL TAIL PROBABILITY

Assuming the factor loadings, α_{ik}, $i = 1, \cdots, n$, $k = 1, \cdots, d$ are all nonnegative,

- The mapping

$$y_k \longmapsto P(L > x | Y_1 = y_1, Y_2 = y_2, \ldots, Y_d = y_d), \quad k = 1, \cdots, d$$

  is non-increasing in y_k.

$$\Downarrow$$

$$\forall \mathbf{Y^d} \in [a_1, b_1] \times [a_2, b_2] \ldots \times [a_d, b_d]$$

$$P(L > x | b_1, \ldots b_d) \leq P\left(L > x | \mathbf{Y^d}\right) \leq P(L > x | a_1, \ldots a_d)$$

- $P(L > x | Y_1, Y_2, \ldots, Y_d)$ is continuous and differentiable with respect to Y_k, $k = 1, \cdots, d$.

A Gaussian one-factor example

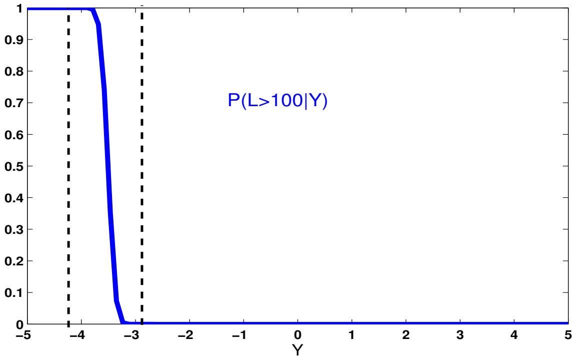


FIGURE: The integrand $P(L > 100|Y)$ as a function of the common factor Y for portfolio A, which consists of 1000 obligors with $w_i = 1$, $p_i = 0.0033$ and $\alpha_i = \sqrt{0.2}$, $i = 1, \ldots, 1000$.

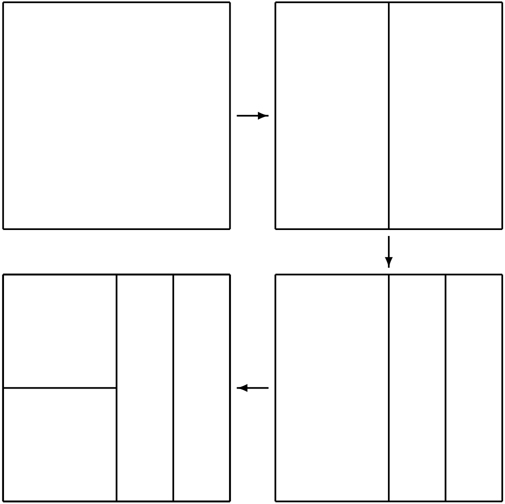
The Gaussian multi-factor model

$$I(f) = P(L > x) = \int \cdots \int f(Y_1, \dots Y_d)\phi(Y_1, \dots Y_d) dY_1 \dots dY_d,$$

where $f(Y_1, \dots Y_d) = P(L > x | Y_1, \dots Y_d)$.

- curse of dimensionality: The product quadrature rule becomes impractical because the number of function evaluations grows exponentially with d.
- (quasi-) Monte Carlo methods: sample uniformly in the cube $[0, 1]^d$.
- focus on the subregions where the integrand is most irregular $\Rightarrow$ adaptive integration.

GLOBALLY ADAPTIVE ALGORITHMS



integration rule

error estimate

Globally adaptive algorithms for numerical integration

1. Choose a subregion from a collection of subregions and subdivide the chosen subregion.
2. Apply an integration rule to the resulting new subregions; update the collection of subregions.
3. Update the global integral and error estimate; check whether a predefined termination criterion is met; if not, go back to step 1.

THE GENZ-MALIK RULE

- A polynomial interpolatory rule of degree 7, which integrates exactly all *monomials* $x_1^{k_1} x_2^{k_2} \dots x_n^{k_d}$ with $\sum k_i \leq 7$ and fails to integrate exactly at least one monomial of degree 8.
- All integration nodes are inside integration domain.
- Requires $2^d + 2d^2 + 2d + 1$ integrand evaluations for a function of d variables, most advantageous for problems with $d \leq 8$. Gauss-Legendre quadrature of degree 7 would need 4^d integrand evaluations.
- A degree 5 rule embedded in the degree 7 rule is used for error estimation, no additional integrand evaluations are necessary.

$$\varepsilon = |I_7 - I_5|.$$

THE GENZ-MALIK RULE

- Bounded integral in each subregion.
  $\forall \mathbf{Y^d} \in [a_1, b_1] \times [a_2, b_2] \ldots \times [a_d, b_d]$

$$\mathscr{L} \leq P\left(L > x \,|\, \mathbf{Y^d}\right) \leq \mathscr{U} \Rightarrow$$

$$\mathscr{L} \prod_{i=1}^{d} (\Phi(b_i) - \Phi(a_i)) \leq I(f) \leq \mathscr{U} \prod_{i=1}^{d} (\Phi(b_i) - \Phi(a_i)).$$

- Local bounds aggregate to a global upper bound and a global lower bound for the whole integration region.
- Asymptotic convergence: $I_7 \rightarrow I(f)$ if we continue with the subdivision until the global upper bound and lower bound coincide.
- Error estimate not so reliable, cf. Lyness & Kaganove (1976), Berntsen (1989).

Adaptive Monte Carlo integration

- Globally adaptive algorithm using Monte Carlo simulation as a basic integration rule.
- Asymptotically convergent.
- Unbiased estimate for the tail probability.
- Practical variance estimate, probabilistic error bounds available.
- Error convergence rate at worst $O\left(1/\sqrt{N}\right)$.
- Number of sampling points in each subregion independent of number of dimensions d.

A 2D EXAMPLE

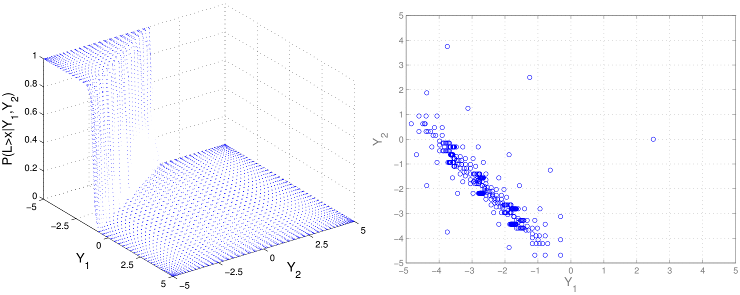


FIGURE: Adaptive Genz-Malik rule for a 2 factor model. (left) integrand $P(L > x | Y_1, Y_2)$; (right) centers of the subregions generated by adaptive integration.

A FIVE-FACTOR MODEL

- 1000 obligors with $w_i = 1$, $p_i = 0.0033$, $i = 1, \ldots, 1000$, grouped into 5 buckets of 200 obligors.
- Factor loadings

$$\alpha_i = \begin{cases} \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), i = 1, \ldots, 200, \\ \left(\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right), i = 201, \ldots, 400, \\ \left(\frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}, \frac{1}{\sqrt{4}}, 0, 0\right), i = 401, \ldots, 600, \\ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0, 0, 0\right), i = 601, \ldots, 800, \\ \left(\frac{1}{\sqrt{2}}, 0, 0, 0, 0\right), i = 800, \ldots, 1000. \end{cases}$$

- Benchmark: plain MC with hundreds of millions of scenarios.

A FIVE-FACTOR MODEL: ADAPTIVE GM

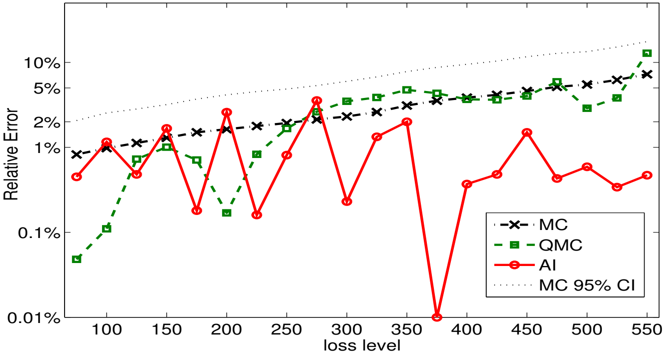


FIGURE: Estimation relative errors of adaptive GM, plain MC and quasi-MC methods with around $N = 10^6$ evaluations for various loss levels.

A five-factor model: Adaptive MC

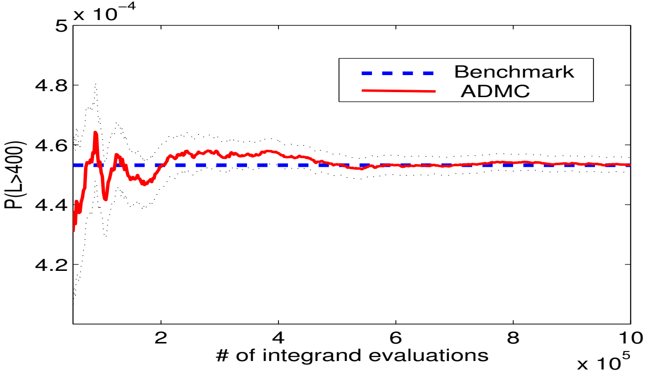
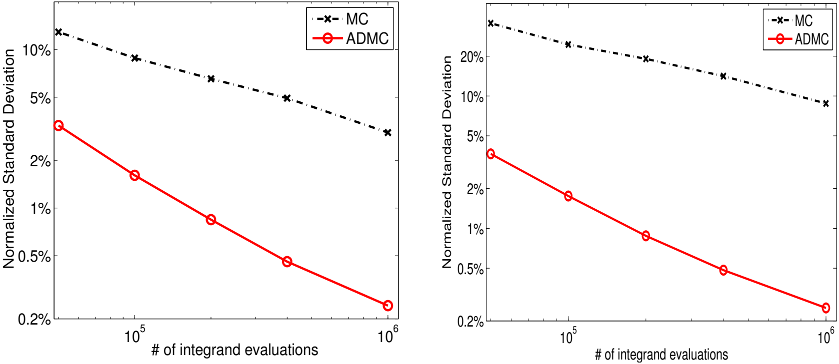


FIGURE: Tail probability $P(L > 400)$ computed by adaptive MC integration with number of integrand evaluations ranging from $50,000$ to 10^6 and their corresponding 95% confidence intervals (dotted lines). The dashed line is our Benchmark.

A FIVE-FACTOR MODEL: ADAPTIVE MC



(a) $x = 300$ (b) $x = 550$

FIGURE: Relative estimation error of $P(L > x)$ by Adaptive MC and plain MC for different loss levels x.

Conclusions

For the calculation of the tail probability in multi-factor portfolio credit loss models,

- Adaptive algorithms are very suitable and particularly attractive for large loss levels.
- Both adaptive Genz-Malik rule and adaptive Monte Carlo integration are asymptotically convergent.
- The adaptive Monte Carlo integration is able to provide practical probabilistic error bounds, with error convergence rate at worst $O\left(1/\sqrt{N}\right)$.

References

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Genz, A. & Malik, A. (1980), ‘An adaptive algorithm for numerical integration over an n-dimensional rectangular region’, Journal of Computational and Applied Mathematics 6(4), 295-302.

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