Set–valued risk measures

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Set–valued risk measures

1. Introduction: From scalar to set–valued risk measures
2. Primal representation: Risk measures and acceptance sets
3. Dual representation: The central role of set–valued expectation
4. Examples
5. One slide about scalarization
6. A selection of open problems

Risk of a scalar position.

- $A \subseteq L^p := L^p(\Omega, \Sigma, P)$, $p \in [0, \infty]$, set of acceptable positions;
- $E \in L^p$ reference instrument with $E(\omega) = 1$ P-a.s.;
- The "risk" of a position $X \in L^p$ is

$$\varrho(X) = \inf \{t \in \mathbb{R} : \ X + tE \in A\},$$

  the minimal number of units of the reference instrument E that has to be added to X in order to get an acceptable position;
- The set

$$R(X) = \{t \in \mathbb{R} : \ X + tE \in A\}$$

  is the set of **all** numbers of units of the reference instrument E that can be added to X in order to get an acceptable position;
- Note: $\mathrm{cl}\left(R(X) + \mathbb{R}_+\right) = \varrho(X) + \mathbb{R}_+$.

Risk of a vector position.

- $A \subseteq L_d^p := L_d^p(\Omega, \Sigma, P)$, $d \geq 1$ set of acceptable positions;
- $E^i \in L_d^p$ reference instrument in market $i \in \{1, \ldots, d\}$ with

$$E^1 = \begin{pmatrix} E \\ 0 \\ \vdots \\ 0 \end{pmatrix}, E^2 = \begin{pmatrix} 0 \\ E \\ \vdots \\ 0 \end{pmatrix}, \ldots, E^d = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ E \end{pmatrix};$$

- Look for linear combinations of reference instruments that give an acceptable position when added to $X \in L_d^p$:

$$R(X) = \left\{ u \in \mathbb{R}^d \colon X + \sum_{i=1}^{d} u_i E^i \in A \right\};$$

- What about $\varrho(X) = \inf \left\{ u \in \mathbb{R}^d \colon X + \sum_{i=1}^{d} u_i E^i \in A \right\}$?

Risk of a vector position.

- Investor/regulator only accepts reference instruments in market $1, \ldots, m$ with $1 \leq m \leq d$:

$$R(X) = \left\{ u \in \mathbb{R}^m \colon X + \sum_{i=1}^{m} u_i E^i \in A \right\}$$

- **Question:** How shall we compare
  - * positions $X^1, X^2 \in L^p_d$?
  - * values $R\left(X^1\right), R\left(X^2\right)$?

- **Answer:** By means of convex cones $K \subseteq \mathbb{R}^d$, $K_m \subseteq \mathbb{R}^m$:
  - * K gives order for X's via $C := \left\{ X \in L^p_d \colon X(\omega) \in K \ P\text{-a.s.} \right\}$
  - * K_m generates order in $\mathbb{R}^m$ and image spaces.

Data and definitions.

- $K \subseteq \mathbb{R}^d$ convex cone (models exchange/transaction rates): If $x \in K$ then $\sum_{i=1}^d x_i E^i$ can be exchanged into a position with non-negative entries only. Reasonable: $\mathbb{R}^d_+ \subseteq K$.
- $K_m = \left\{u \in \mathbb{R}^m \colon (u_1, \ldots, u_m, 0, \ldots, 0)^T \in K\right\}$. Then $\mathbb{R}^m_+ \subseteq K_m$.
- Image spaces

  $\mathcal{F}_m := \{M \subseteq \mathbb{R}^m : \ M = \mathrm{cl}\,(M + K_m)\}$,

  $\mathcal{C}_m := \{M \subseteq \mathbb{R}^m : \ M = \mathrm{cl\,co}\,(M + K_m)\}$;
- $R \colon L^p_d \to \mathcal{F}_m$ is convex (sublinear, closed) iff $\mathrm{epi}\, R$ is convex (a convex cone, a closed set) with

$$\mathrm{epi}\, R := \left\{(X, u) \in L^p_d \times \mathbb{R}^m \colon u \in R(X)\right\}.$$

Set-valued measure of risk. Function $R : L_d^p \to \mathcal{F}_m$:

(R0) normalized, i.e. $K_m \subseteq R(0)$ and $R(0) \cap -\operatorname{int} K_m = \emptyset$;

(R1) translative w.r.t. $E^1, \dots, E^m \in \left(L_d^p\right)_+$, i.e.

$$\forall X \in L_d^p,\ \forall u \in \mathbb{R}^m:\ R\left(X + \sum_{i=1}^m u_i E^i\right) = R(X) + \{-u\};$$

(R2) C–monotone, i.e., $X^2 - X^1 \in C$ implies $R\left(X^2\right) \supseteq R\left(X^1\right)$.

If R satisfies (R0), (R1), (R2) and is convex then it is called a convex measure of risk ($R : L_d^p \to \mathcal{C}_m$ in this case).

If R satisfies (R0), (R1), (R2) and is sublinear then it is called a coherent measure of risk ($R : L_d^p \to \mathcal{C}_m$ in this case).

Acceptance set. Subset $A \subseteq L^p_d$:

(A0) $u \in K_m \Rightarrow \sum_{i=1}^m u_i E^i \in A$; $u \in -\operatorname{int} K_m \Rightarrow \sum_{i=1}^m u_i E^i \notin A$;

(A1) A is radially closed; $u \in K_m \Rightarrow A + \left\{\sum_{i=1}^m u_i E^i\right\} \subseteq A$;

(A2) $A + C \subseteq A$.

If A satisfies (A0), (A1), (A2) and is convex then it is called a convex acceptance set.

If A satisfies (A0), (A1), (A2) and is a convex cone then it is called a coherent acceptance set.

Radially closed w.r.t. $E^1, \ldots, E^m$:

$X \in L^p_d$, $\left\{u^k\right\}_{k \in \mathbb{N}} \subset \mathbb{R}^m$, $\lim_{k\to\infty} u^k = 0$, $\forall k \in \mathbb{N}: \; X + \sum_{i=1}^m u_i^k E^i \in A$ $\Rightarrow X \in A$.

Primal representation result. $R\colon L^p_d \to \mathcal{F}_m,\ A \subseteq L^p_d$

$$A_R := \left\{X \in L^p_d :\ K_m \subseteq R(X)\right\}$$

$$R_A(X) := \left\{u \in \mathbb{R}^m :\ X + \textstyle\sum_{i=1}^m u_i E^i \in A\right\}$$

Theorem. (i) Let $R : L^p_d \to \mathcal{F}_m$ be a measure of risk. Then A_R is an acceptance set and $R = R_{A_R}$. If R is convex, so is A. If R is coherent then A is a coherent acceptance set.

(ii) Let $A \subseteq L^p_d$ be an acceptance set. Then R_A is a measure of risk and $A = A_{R_A}$. If A is convex, so is R_A. If A is a coherent acceptance set then R_A is a coherent measure of risk.

Scalar coherent risk measure. $\varrho \colon L^p \to \mathbb{R} \cup \{+\infty\}$

$$\varrho(X) = \sup_{Q \in \mathcal{Q}} E^Q[-X]$$

with $\mathcal{Q}$ a set of probability measures, abs. cont. w.r.t. P.

Set–valued coherent risk measure. $R : L^p_d \to \mathcal{C}_m$

$$R(X) = \sup_{Q \in \mathcal{Q}} ??$$

Question: What is set–valued $E^Q[-X]$??

Set–valued expectation. $1 \leq p < \infty$ (case $p = \infty$ parallel)

$$C^+ = \left\{ Z \in L^q_d \colon \forall X \in C \colon \int_\Omega X \cdot Z \, dP \geq 0 \right\}, \; \frac{1}{p} + \frac{1}{q} = 1$$

$$\mathcal{Z}^q_m = \left\{ Z \in C^+ \colon \sum_{i=1}^m E^P \left[Z_i \right] = 1 \right\}$$

$$F^Z_m \left[X \right] = \left\{ u \in \mathbb{R}^m : \int_\Omega \left(X - \sum_{i=1}^m u_i E^i \right) \cdot Z \, dP \leq 0 \right\}.$$

- If $m = d = 1$ then $F^Z_m \left[X \right] = E^Q \left[X \right] + \mathbb{R}_+$ with $\frac{dQ}{dP} = Z$.
- $Z \in \mathcal{Z}^q_m \Rightarrow R(X) = F^Z_m \left[-X \right]$ is a coherent risk measure on L^p_d.

Theorem. $R : L_d^p \to \mathcal{C}_m$ proper closed convex measure of risk:

$$\forall X \in L_d^p : \; R(X) = \bigcap_{Z \in \mathcal{Z}_m^q} \left(F_m^Z[-X] + \text{cl} \bigcup_{X' \in A_R} F_m^Z\left[X'\right] \right).$$

R additionally positively homogeneous:

$$\forall X \in L_d^p : \; R(X) = \bigcap_{Z \in \mathcal{Z}_m^q \cap A_R^+} F_m^Z[-X].$$

Recall. $\varrho(X) = \sup_{Q \in \mathcal{Q}} \left(E^Q[-X] - \sup_{X' \in A_\varrho} E^Q\left[X'\right] \right)$

Dual summary.

- Basic rule: Replace $E[\cdot]$ by $F_m^Z[\cdot]$!
- Basic result: Everything as in the extended real–valued case like
    - Penalty function representation (Föllmer/Schied)
    - L_d^1–representation of weak* closed risk measures on L_d^∞
    - "dual" ways of defining convex risk measures
- Basic tool: Duality theory for set–valued convex functions

Summary of the summary.

Everything you can do scalar you can do set–valued!!!

4.1. Set–valued expectation. See above.

4.2. Set–valued componentwise expectation. $1 \leq p \leq \infty$

$$A := \left\{X \in L^p_d : \ E^P[X] \in K\right\}, \quad CE(X) := R_A(X)$$

are coherent with $CE(X) = \left\{u \in \mathbb{R}^m : \ E^P\left[X + \sum_{i=1}^m u_i E^i\right] \in K\right\}$.

4.3. Set–valued essential infimum. Coherent case

$$-EI^S(X) = \left\{u \in \mathbb{R}^m : \ X + \sum_{i=1}^m u_i E^i \in C\right\} =$$

$$\left\{u \in \mathbb{R}^m : \ P\left(\left\{\omega \in \Omega : \ X(\omega) + \sum_{i=1}^m u_i E^i(\omega) \notin K\right\}\right) = 0\right\}.$$

4.4. Set–valued V@R. $0 \leq \lambda \leq 1$, strong variant

$$V@R_\lambda^S(X) = \left\{u \in \mathbb{R}^m : \; P\left(\left\{\omega \in \Omega \colon X(\omega) + \sum_{i=1}^m u_i E^i(\omega) \notin K\right\}\right) \leq \lambda\right\}.$$

4.5. Set–valued V@R. $0 \leq \lambda \leq 1$, a weak variant

$$V@R_\lambda^W(X) = \left\{u \in \mathbb{R}^m \colon P\left(\left\{\omega \in \Omega : \; X(\omega) + \sum_{i=1}^m u_i E^i(\omega) \in -\operatorname{int} K\right\}\right) \leq \lambda\right\}.$$

One can replace $-\operatorname{int} K$ by something bigger not intersecting K!

4.6. Set–valued AV@R. $1 \leq p < \infty$, $0 < \lambda \leq 1$,

$$\mathcal{Z}_\lambda := \left\{ Z \in \mathcal{Z}^q_m : \ \exists v \in \mathbb{R}^m_+ : \sum_{i=1}^m v_i = \frac{1}{\lambda}, \ \forall i = 1, ...m : Z_i \leq v_i E \right\},$$

$$AV@R_\lambda(X) := \bigcap_{Z \in \mathcal{Z}_\lambda} F^Z_m[-X]$$

is coherent on L^p_d. Note: $Z \in \mathcal{Z}^q_m \Rightarrow Z \geq 0$.

4.7. Entropic risk measure. Convex, but not coherent.

$\beta > 0$, $E^{(d)} = (E, \ldots, E)^T \in L_d^\infty$,

$$\begin{aligned} \mathcal{Q}_m &= \left\{ Q \in ba_d : \; Q \in C^+, \; \sum_{i=1}^m \int_\Omega E \, dQ_i = 1 \right\} \\ \widetilde{\mathcal{Q}}_m &= \left\{ Q \in \mathcal{Q}_m : \; \exists \frac{dQ_i}{dP} = Z_i \in L^1, i = 1, \ldots, m \right\} \\ G\left(Q \middle| \, P\right) &:= F_m^Q \left[E^{(d)} \log \left(\sum_{i=1}^m \frac{dQ_i}{dP} \right) \right], \end{aligned}$$

$$R_\beta\left(X\right) := \bigcap_{Q \in \widetilde{\mathcal{Q}}_m} \left[-\frac{1}{\beta} G\left(Q \middle| \, P\right) + F_m^Q\left[-X\right] \right].$$

Example summary.

- If $m = d = 1$ then each of the above examples yields its scalar counterpart.
- Sometimes, there are more and less risk averse set–valued extensions of the same scalar risk measure (ess. infimum, V@R).
- Definitions possible

    * direct

    * via acceptance sets (primal representation)

    * via "penalty functions" (dual representation).

Question: $R(X)$ given. Which $u \in R(X)$ shall I(nvestor) choose?

1. Answer: Choose minimal ("efficient") point w.r.t $\leq_{K_m}$.

2. Answer: (strongly related) Realize value of

$$\varphi_v : L^p_d \to \mathbb{R} \cup \{\pm\infty\}, \quad \varphi_v(X) = \inf_{u \in R(X)} v^T u, \quad v \in K^+_m.$$

Interpretation.

• $v \in K^+_m$ is vector of "reduced prices" for accepted reference instrument

• $\varphi_v(X)$ is minimal price I have to pay for a position of accepted reference instruments that cancels the risk of X.

Result. Commuting diagram for R, φ_v and its Fenchel conjugates $(\varphi_v)^* = \varphi^*_v$, $-R^*$:

Scalarization sceme.

$$
\begin{array}{ccc}
R & \overset{\text{Scalarization}}{\longleftarrow\,\longrightarrow} & \varphi_v \\
\uparrow & & \uparrow \\
\text{Conjugation} & & \text{Conjugation} \\
\downarrow & & \downarrow \\
-R^* & \overset{\text{Scalarization}}{\longleftarrow\,\longrightarrow} & \varphi_v^*
\end{array}
$$

$-R^*\colon L_d^q \times K_m^* \to \mathcal{C}_m$ set–valued Legendre–Fenchel transform of R,

$$\varphi_v^* : L_d^q \to \mathbb{R} \cup \{\pm\infty\}, \quad \varphi_v^*(-Z) = \sup_{u \in -R^*(-Z,-v)} -v^T u,$$

Open problems. (a selection)

- Primal representation of set–valued AV@R?
- More (about) entropic risk measures?
- Optimization problems with set–valued risk measures (capital allocation, portfolio optimization etc.)?
- Relationships between set–valued risk measures, vector optimization and scalarization procedures

$$\longrightarrow \varrho(X) = \text{"inf"}\{u \in \mathbb{R}^m \colon u \in R(X)\}$$

Last slide

And remember: Everything you can do scalar ...

... **thank you very much for attention!**

Selected references.

1. Jouini, E., Meddeb, M., Touzi, N., *Vector-Valued Coherent Risk Measures*, Finance and Stochastics, Vol. 8(4), 531-552, (2004)

2. Cascos, I., Molchanov, I, *Multivariate Risks and Depth-Trimmed Regions*, Finance & Stochastics, Vol. 11, 373-397, (2007)

3. Hamel, A.H., Heyde, F., Höhne, M., *Set–valued Measures of Risk*, submitted, 2007