Non-Monotone Risk Measures and Monotone Hulls

Damir Filipović
(joint with Michael Kupper)

University of Munich
www.math.lmu.de/~filipo

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Overview

- Motivating examples
- Convex monotone cash-invariant functions
- Monotone and cash-invariant hulls
- Examples

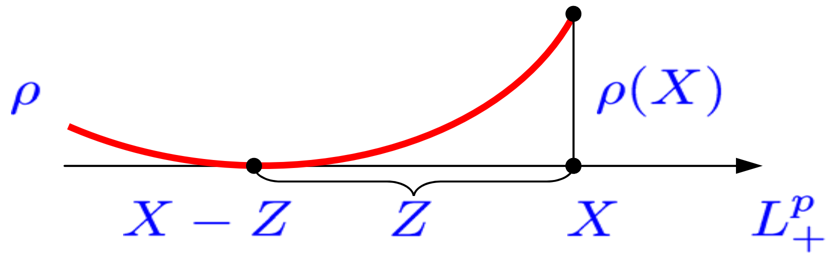
Non-Monotone Risk Measures

Suppose the regulator specifies a **non-monotone** risk measure

$$\rho : L^p \to (-\infty, \infty]$$

for the insurance industry to determine the risk capital requirements.

The management may withdraw a positive portion $Z \geq 0$ of the portfolio X without increasing the capital requirement.



How come?

Example: Swiss Solvency Test Risk Measure

Multi-period risk measure

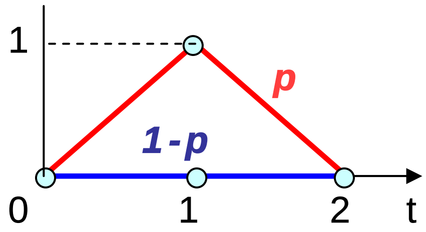
$$\rho(C) = \mathrm{ES}(C_1) + \gamma \sum_{t=2}^{T} \mathrm{ES}(C_t - C_{t-1})$$

where

- $C = (C_0, C_1, \ldots, C_T)$ = asset-liability value process
- ES = expected shortfall (at 99% level)
- γ = cost of capital spread

Example: let $T = 2$, $0.01 < p < 0.99$.

$$\rho(C) = 0 + \gamma \cdot 1 > 0$$



Example: Mean-Variance Risk Measure

Classical framework for asset allocation

$$\rho(X) = \mathrm{E}[-X] + \frac{\alpha}{2}\mathrm{Var}[X]$$

Example: Semi-Moment Risk Measure

$\rho(X) = \frac{1}{\alpha}\mathbb{E}[X_-]$ convex monotone, not cash-invariant

Program

- Characterize convex monotone functions
- Characterize convex cash-invariant functions
- Construct monotone and cash-invariant hulls
- What can we learn from it?
  →Construct new and rediscover old convex risk measures

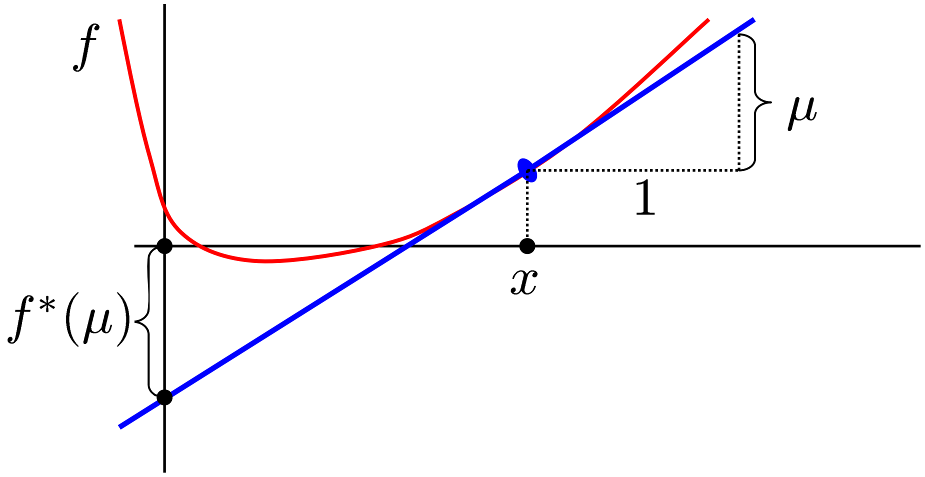
Convex functions

$f : L^p \to (-\infty, +\infty]$ convex, $\quad (L^p)^* = L^q$

The **conjugate** $f^*(\mu) := \sup_{x \in L^p} (\langle \mu, x \rangle - f(x))$ is l.s.c. convex.

If f is l.s.c. then $f(x) = \sup_{\mu \in L^q} (\langle \mu, x \rangle - f^*(\mu))$.

$\mu \in \partial f(x) := \{\nu \in L^q \mid f(y) \geq f(x) + \langle \nu, y - x \rangle\}$ (**subgradients**)
if and only if $f(x) = \langle \mu, x \rangle - f^*(\mu)$:

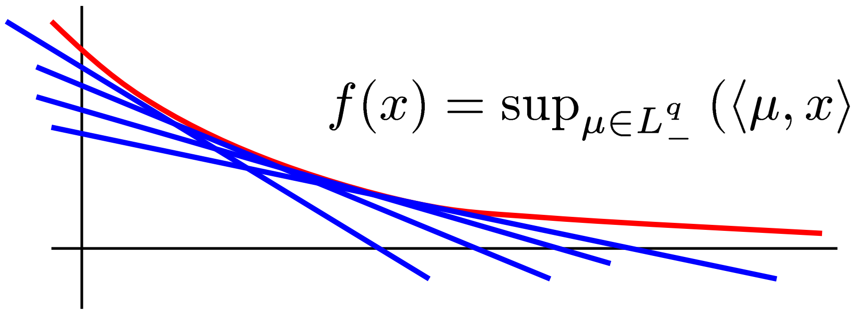


Monotone functions

Polar cone $(L^p_+)^\circ = L^q_-$

Definition: f is **monotone** if $f(X) \le f(Y)$ for all $X \ge Y$.

Lemma: $f : L^p \to (-\infty, +\infty]$ l.s.c. convex is monotone if and only if $\mathrm{dom}(f^*) \subset L^q_-$.



$$f(x) = \sup_{\mu \in L^q_-} \left(\langle \mu, x \rangle - f^*(\mu) \right)$$

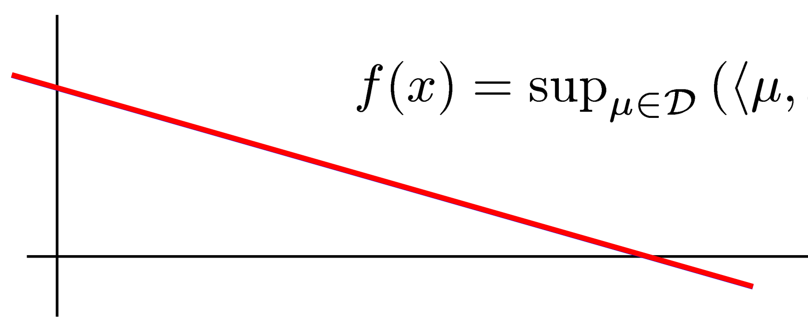
Cash-invariant functions

Numeraire (“cash”) $\Pi \equiv 1$ (w.l.o.g.)

Definition: f is **cash-invariant** if $f(x + c\Pi) = f(x) - c$, $\forall c \in \mathbb{R}$.

Define $\mathcal{D} := \{\mu \in L^q \mid \langle \mu, \Pi \rangle = -1\}$ (normalized elements)

Lemma: $f : L^p \to (-\infty, +\infty]$ l.s.c. convex is cash-invariant if and only if $\operatorname{dom}(f^*) \subset \mathcal{D}$



$$f(x) = \sup_{\mu \in \mathcal{D}} \left(\langle \mu, x \rangle - f^*(\mu) \right)$$

Maximal elements

Define

$$\delta(x \mid \mathcal{C}) := \begin{cases} 0, & x \in \mathcal{C}; \\ +\infty, & x \notin \mathcal{C}. \end{cases} = \textbf{indicator function} \text{ of a set } \mathcal{C}$$

$$\delta^*(\mu \mid \mathcal{C}) = \sup_{x \in \mathcal{C}} \langle \mu, x \rangle = \textbf{support function} \text{ of } \mathcal{C}$$

Lemma: Among the l.s.c. convex functions vanishing at $x = 0$,

1. $\delta(\cdot \mid L^p_+) = \delta^*(\cdot \mid L^q_-)$ is the greatest monotone;
2. $\delta^*(\cdot \mid \mathcal{D})$ is the greatest cash-invariant;
3. $\delta^*(\cdot \mid L^q_- \cap \mathcal{D}) = -\operatorname{ess\,inf}(\cdot)$ is the greatest monotone cash-invariant.

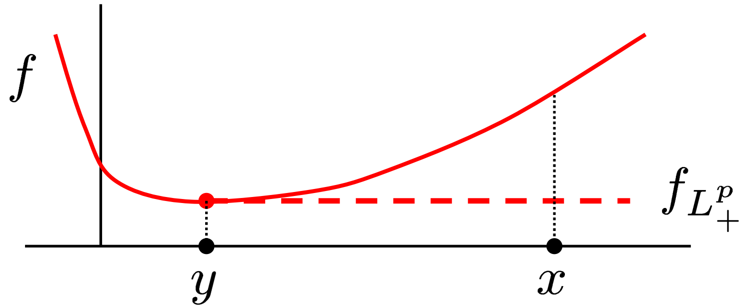
Monotone hulls

Infimal convolution $f \Box g(X) := \inf_{Z \in L^p} f(X - Z) + g(Z)$

Definition: The **monotone hull** of f is

$$f_{L^p_+}(X) := \inf_{Z \geq 0} f(X - Z) = f \Box \delta(\cdot \mid L^p_+)(X).$$

Theorem: $f_{L^p_+}$ is monotone with $f_{L^p_+} \leq f$, and $f_{L^p_+} = f$ if and only if f is monotone. Moreover, $f^*_{L^p_+} = f^* + \delta(\cdot \mid L^q_-)$ and $f^{**}_{L^p_+}$ is the greatest l.s.c. convex monotone function majorized by f.



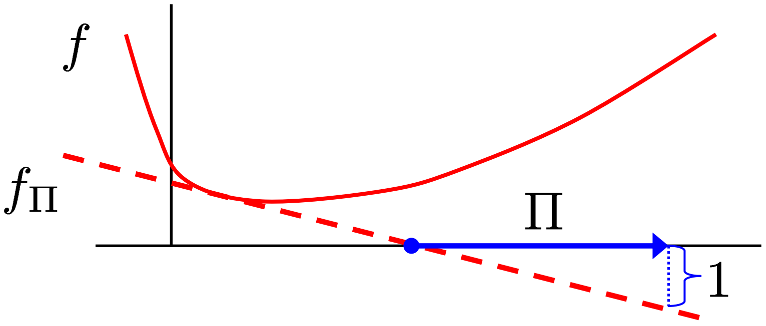
Cash-invariant hulls

Lemma: $\delta^*(X \mid \mathcal{D}) = \begin{cases} -\lambda, & \text{if } X = \lambda\Pi, \\ +\infty, & \text{else.} \end{cases}$

Definition: The **cash-invariant hull** of f is

$$f_\Pi(X) := f \Box \delta^*(\cdot \mid \mathcal{D})(X) = \inf_{\lambda\in\mathbb{R}} f(X - \lambda\Pi) - \lambda$$

Theorem: f_Π is cash-invariant with $f_\Pi \leq f$, and $f_\Pi = f$ if and only if f is cash-invariant. Moreover, $f_\Pi^* = f^* + \delta(\cdot \mid \mathcal{D})$, and f_Π^{**} is the greatest l.s.c. convex cash-invariant function majorized by f.



Monotone cash-invariant hulls

Theorem: $\delta^*(\cdot \mid \mathcal{D}) \Box \delta^*(\cdot \mid L^q_-) = \delta^*(\cdot \mid L^q_- \cap \mathcal{D})$. Hence

$$f_{L^p_+,\Pi}(X) = \inf_{Y \in L^p} \left(f(X - Y) - \operatorname{ess\,inf} Y \right)$$

Recipe: $f^\sharp := \big(f^* + \delta(\cdot \mid L^q_- \cap \mathcal{D})\big)^*$ is the greatest l.s.c. convex monotone cash-invariant function majorized by f

Mean-Variance Risk Measure

$f(X) = \mathbb{E}[-X] + \frac{\alpha}{2}\mathbb{E}[X^2]$ not cash-invariant, not monotone

Cash-invariant hull of f is the **mean-variance risk measure**:

$$f_\Pi(X) = \inf_{\lambda\in\mathbb{R}} \left(\mathbb{E}[-X] + \frac{\alpha}{2}\mathbb{E}[|X-\lambda|^2]\right) = \mathbb{E}[-X] + \frac{\alpha}{2}\|(X - \mathbb{E}[X])\|_2^2.$$

Calculation of $f^\sharp$: write $g(X) := \frac{\alpha}{2}\mathbb{E}[X^2]$, so that

$$f^*(Z) = \sup_{X\in L^2} \left(\mathbb{E}[(Z+1)X] - g(X)\right) = g^*(Z+1).$$

We have $g^*(Z) = \sup_{X\in L^2} \mathbb{E}\left[ZX - \frac{\alpha}{2}X^2\right] = \frac{1}{2\alpha}\mathbb{E}[Z^2]$ and thus

$$f^\sharp(X) = \sup\left\{\mathbb{E}[-ZX] - \frac{1}{2\alpha}\mathbb{E}[(Z-1)^2] \mid Z \in L^2_+,\ \mathbb{E}[Z]=1\right\}$$

monotone mean-variance risk measure (→ Maccheroni et al (2005))

Semi-Moment Risk Measure

$f(X) = \frac{1}{\alpha}\mathbb{E}[X_-]$ convex monotone, not cash-invariant

Monotone cash-invariant hull of f is

$$f_\Pi(X) = \inf_{\lambda\in\mathbb{R}} \left(\frac{1}{\alpha}\mathbb{E}[(X-\lambda)_-] - \lambda \right) = \frac{1}{\alpha}\mathbb{E}[(X-\hat{\lambda})_-] - \hat{\lambda}$$

where optimizer $\hat{\lambda}$ satisfies $\mathbb{P}[X < \hat{\lambda}] \le \alpha \le \mathbb{P}[X \le \hat{\lambda}]$; i.e. $\hat{\lambda}$ is an α-quantile of X. Hence

$$f_\Pi(X) = ES_\alpha[X]$$

$\Rightarrow$ **expected shortfall** is the cash-invariant hull of f

Exponential Risk Measure

$f(X) = \mathbb{E}[\exp(-X)] - 1$ convex monotone, not cash-invariant

optimizer $\hat{\lambda} = -\log \mathbb{E}[\exp(-X)]$

Monotone cash-invariant hull of f is

$$f_\Pi(X) = \inf_{\lambda \in \mathbb{R}} (\mathbb{E}[\exp(-X + \lambda) - 1 - \lambda) = \log \mathbb{E}[\exp(-X)]$$

$\Rightarrow$ **entropic risk measure** is the cash-invariant hull of f

Generalized: $f(X) = \mathbb{E}[g(X)]$ with $g : \mathbb{R} \to \mathbb{R}$ convex, $g(0) = 0$

Swiss Solvency Test Risk Measure

$\rho(C) = \mathrm{ES}(C_1) + \gamma \sum_{t=2}^T \mathrm{ES}(\Delta C_t)$ not monotone

Write $\rho(C) = f(C_1, \Delta C_2, \ldots, \Delta C_T)$ for $f(X) = \mathrm{ES}(X_1) + \gamma \sum_{t=2}^T \mathrm{ES}(X_t)$ on model space $E = \prod_{t=1}^T L^p(\mathcal{F}_t)$ with dual $E^* = \prod_{t=1}^T L^q(\mathcal{F}_t)$

Order cone $\mathcal{P} = \{X \in E \mid \sum_{s=1}^t X_s \geq 0 \,\forall t \leq T\}$
Polar cone $\mathcal{P}^\circ = \{\mu \in E^* \mid \mathbb{E}[\mu_t - \mu_{t+1} \mid \mathcal{F}_t] \leq 0 \,\forall t \leq T\}$ $(\mu_{T+1} := 0)$

Define $\mathcal{M}_t := \{\mu \in L^q(\mathcal{F}_t) \mid \mathbb{E}[\mu] = -1 \text{ and } -1/\alpha \leq \mu \leq 0\}$
Then $f^*(\mu) = \delta(\mu_1 \mid \mathcal{M}_1) + \sum_{t=2}^T \delta(\mu_t \mid \gamma \mathcal{M}_t)$
Hence $\mathrm{dom} f^* = \mathcal{M}_1 \times \gamma \prod_{t=2}^T \mathcal{M}_t$

Lemma: $\mu \in \mathrm{dom} f^* \cap \mathcal{P}^\circ$ if and only if $\mu_1 = (1-\gamma)\nu_1 + \gamma \mathbb{E}[\nu_T \mid \mathcal{F}_1]$ and $\mu_t = \gamma \mathbb{E}[\nu_T \mid \mathcal{F}_t]$ for $t \geq 2$, for some $\nu_1 \in \mathcal{M}_1$, $\nu_T \in \mathcal{M}_T$.

Hence $f^\sharp(X) = \sup_{\mu \in \mathrm{dom} f^* \cap \mathcal{P}^\circ} \langle \mu, X \rangle = (1-\gamma)\mathrm{ES}(X_1) + \gamma \mathrm{ES}(\sum_t X_t)$

$\Rightarrow$ **Monotone Hull** $\rho_{\mathcal{P}}(C) = (1-\gamma)\mathrm{ES}(C_1) + \gamma \mathrm{ES}(C_T)$

Conclusion

- We found tractable ways to explicitly calculate monotone and cash-invariant hulls.

→ Fix defective risk measures

→ Construct new ones

- Thanks!