Dynamic Correlation Modeling

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September 21, 2007

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1. Motivation

- Deficiencies of the copula approach:
  1. Unable to *explain* market quotes: correlation structure imposed exogenously and in a arbitrary fashion.
  2. Inconsistency across time/maturities.
  3. Lack of dynamics: not suitable to price exotic products such as option on tranche spreads.
- Ideally we would like to employ a model which is:
  1. Flexible enough to fit market data (i.e. calibrate to standard tranche spreads across the capital structure and all liquid maturities).
  2. Dynamically consistent.
  3. Tractable and computationally efficient.
- The key idea behind the dynamic approach developed by Di Graziano and Rogers is to use a stochastic **process** (and not a **random variable** as in the traditional factor model approach) to drive the common dynamics of the various credits in the portfolio.
- In order to retain tractability and computational efficiency the process chosen to drive the portfolio common dynamics is a continuous time Markov chain.

2. ...Some standard result about continuous time Markov chains

- Let $(\xi_t)_{t\geq 0}$ be a continuous time K-dimensional, Markov chain with infinitesimal generator Q taking values in the set $\mathcal{K} \equiv \{1, 2, \ldots, K\}$
- Transition probabilities are given by:

$$P_{ij}(t) \equiv P(\xi_{t+s} = j \mid \xi_t = i) = \left(e^{Qt}\right)_{ij} \tag{1}$$

- Since the process ξ_t takes values in the finite set $\mathcal{K}$, any function of the chain, $f : \mathcal{K} \to \mathbb{R}$ can be represented as a K-dimensional vector with i^{th} component given by $f_i \equiv f(i)$, i.e.

$$\begin{pmatrix} f_1 \\ f_2 \\ \cdots \\ \cdots \\ f_K \end{pmatrix} \tag{2}$$

- Let $\alpha(\xi)$ and $f(\xi)$ be K dimensional vectors, and let $J_{ik}(t)$ represent the number of jumps from state i to state j up to time t. We have that

$$\begin{aligned} V_t(\xi) &\equiv E\left[\exp\left(-\int_0^t \alpha(\xi_u)du - \sum_{i\neq j} w_{ij} J_{ij}(t)\right) f(\xi_t) \mid \xi_0 = \xi\right] \\ &= \left(e^{\tilde{Q}t} f\right)(\xi) \end{aligned}$$

where

$$
\begin{aligned}
\tilde{Q}^i_{jk} &= Q_{jj} - \alpha_j && (j = k); \\
&= \exp(-w_{jk}) Q_{jk} && (j \neq k).
\end{aligned}
$$

3. Model set up

- The **common dynamics** of the names in the porfolio are driven by a continuous time, K-dimensional, Markov chain $(\xi_t)_{t\geq 0}$ with infinitesimal generator (Q-matrix) Q.
- Conditional on the process ξ, i.e. on $\mathcal{F}_t^\xi \equiv \sigma(\xi_s, s \leq t)$ default times τ_i are independent.
- The key modeling ingredient of our approach is given by the conditional survival probability of the single names

$$q_t^i = P\left(\tau^i \geq t \mid F_t^\xi\right) = \exp\left(-C_t^i\right), \tag{3}$$

where C_t^i is some additive functional of the chain of the form

$$C_t^i = \int_0^t \lambda^i(\xi_u)du + \sum_{j\neq k} w_{jk}^i J_{jk}(t). \tag{4}$$

.

- The short rate is assumed to be a function of the chain. The discount factor is given by

$$B_t^{-1} \equiv \exp\left(-\int_0^t r(\xi_u)du\right). \tag{5}$$

- The survival probability is given by

$$\begin{aligned} q_t^i(\xi_0) &\equiv E\left[1_{\{\tau^i\geq t\}} \mid \xi_0\right] \\ &= \exp(t\tilde{Q}^i)\mathbf{1}(\xi_0), \end{aligned} \tag{6}$$

where

$$\tilde{Q}^i_{jk} = Q_{jj} - \lambda^i_j \qquad (j = k); \tag{7}$$

$$= Q_{jk} \exp(-w_{jk}) \qquad (j \neq k). \tag{8}$$

4. CDS

- The CDS's *Default leg* can be calculated explicitly as,

$$
\begin{aligned}
DL_T &\equiv E\left[B_\tau^{-1}; \tau \leq T\right] && (9)\\
&= E\left[\int_0^T \{\lambda(\xi_u) + \sum_k Q_{\xi_u k}\theta_{\xi_u k}\} B_u^{-1}\exp(-C_u)du\right] && (10)\\
&= \hat{Q}^{-1}(\exp(\hat{Q}T) - I)\tilde{\lambda}(\xi_0) && (11)
\end{aligned}
$$

where $\tilde{\lambda}_i = \lambda_i + \sum_k Q_{ik}\theta_{ik}$.

- The *PV01* of the CDS is given by

$$
\begin{aligned}
PL_T &\equiv E\left[\int_0^T I_{\{\tau > u\}} B_u^{-1} du\right] = \int_0^T \exp(u\hat{Q})\mathbf{1}du\\
&= \hat{Q}^{-1}(\exp(\hat{Q}T) - I)\mathbf{1}(\xi_0)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{Q}^i_{jk} &= Q_{jj} - r_j - \lambda^i_j \qquad (j = k);\\
&= \exp(-w^i_{jk})Q_{jk} \qquad (j \neq k).
\end{aligned}
$$

- The above calculations allow us to calibrate our model to single name CDS and index spreads.

5. Survival correlation

- Default and survival correlations can be calculated explicitly in this model.
- The pairwise joint survival probability of name i and j is

$$\begin{aligned}\tilde{q}_T^{ij}(\xi_t) &\equiv P(\tau^i \geq T, \tau^j \geq t \mid \xi_t) \\ &= \exp(\tilde{Q}^{ij}(T-t))(\xi_t),\end{aligned}$$

where

$$\begin{aligned}\tilde{Q}_{kl}^{ij} &= Q_{kk} - \lambda_k^i - \lambda_k^j \qquad (k=l); \\ &= \exp(-w_{kl}^i - w_{kl}^j)Q_{kl} \qquad (k \neq l).\end{aligned}$$

- ...and the survival correlation at time t for maturity T is

$$\rho_T(\xi_t) = \frac{\tilde{q}_T^{ij}(\xi_t) - \tilde{q}_T^i(\xi_t)\tilde{q}_T^j(\xi_t)}{\sqrt{\tilde{q}_T^i(\xi_t)(1-\tilde{q}_T^i(\xi_t))}\sqrt{\tilde{q}_T^j(\xi_t)(1-\tilde{q}_T^j(\xi_t))}} \tag{12}$$

where

$$\tilde{q}_T^i(\xi_t) = \exp(\tilde{Q}^i(T-t))(\xi_t). \tag{13}$$

- In this set-up, the survival (default) correlation is a stochastic process driven by ξ. We are in front of a dynamic correlation approach.
- Note that the correlation of defaults is obtained endogenously from the model, rather than being exogenously imposed as in the copula approach.

6. Portfolio loss distribution

- We defined the portfolio loss distribution as

$$L_t \equiv \sum_{i=1}^{N} \ell_i I_{\{\tau_i \le t\}}. \tag{14}$$

- Let now the loss at default of the i^{th} reference entity be given by $l_i = A_i(1 - R_i)$.
- The Laplace transform of the (discounted) loss process is

$$\begin{aligned} E\exp(-\int_0^t r(\xi_s)ds - \alpha L_t) &= E\exp(-\int_0^t r(\xi_s)ds - \alpha \sum_{i=1}^{N} \ell_i I_{\{\tau_i \le t\}}) && (15) \\ &= E\left[\exp(-\int_0^t r(\xi_s)ds) \prod_{i=1}^{N} E\left[e^{-\alpha \ell_i 1_{\{\tau^i \le t\}}} \mid \mathcal{F}_t^{\xi}\right]\right] && (16) \\ &= E\left[\exp(-\int_0^t r(\xi_s)ds) \prod_{i=1}^{N} \left((1 - q_t^i)\zeta_i(\alpha) + q_t^i\right)\right] && (17) \end{aligned}$$

where $\zeta_i(\alpha) = Ee^{-\alpha \ell_i}$ and

$$q_t^i \equiv \exp\left(-\int_0^t \lambda^i(\xi_u)du - \sum_{j \ne k} w_{jk}^i J_{jk}(t)\right). \tag{18}$$

- Equation (17) allows us to derive the law of L_t and to price a range of multi-name credit derivatives.

7. Computational approaches

- How do we compute (17)?

$$\begin{cases} 1. & \text{Exact method;} \\ 2. & \text{Poisson approximation;} \\ 3. & \text{Monte Carlo.} \end{cases} \tag{19}$$

- **Exact method:** multiply out the product on the RHS of (17). The individual terms of the resulting sum are exponentials of some additive functional of the chain, and can be computed explicitly. However for large portfolios this method is inefficient as we need to sum over 2^N terms.

- **Poisson approximation:**

- Conditional on the path of the chain, L_t is approximately compound Poisson with parameter

$$\Lambda_t = \sum_{i=1}^{N} \left(1 - \exp(-C_t^i)\right) \approx \sum_{i=1}^{N} C_t^i \tag{20}$$

- The discounted Laplace transform of L_t can be approximated by

$$\begin{aligned} E \exp(-\int_0^t r(\xi_s)ds - \alpha \bar{L}_t) &= E \exp\left(-\int_0^t r(\xi_s)ds + \sum_{i=1}^{N}(\zeta_i(\alpha) - 1)C_t^i\right) \\ &= e^{\bar{Q}T}\mathbf{1}(\xi_0), \end{aligned}$$

where

$$\begin{aligned} \bar{Q}_{jk} &= Q_{jj} - \nu_j \qquad (j = k); \\ &= \exp(-w_{jk})Q_{jk} \qquad (j \neq k). \end{aligned}$$

where

$$\begin{aligned} \nu &\equiv r + \sum_{i=1}^{N} (1 - \zeta_i(\alpha))\lambda^i \\ w_{jk} &\equiv \sum_{i=1}^{N} w_{jk}^i \end{aligned}$$

- ...which is a simple and rapid calculation.
- **Monte Carlo**
- Calculating (17) boils down to simulating the path of the chain up to T, the maturity of the claim:
  1. Let i be the current state of the chain. Generate an exponential(1) random variable z,
  2. let τ denote the time elapsed from the last jump: set $\tau = z/q_i$,
  3. if $\tau \geq T$, stop otherwise go to step 4,.
  4. sample $\xi(\tau)$ according to probabilities (q_{ij}/q_i), where $j \neq i$ and $j \leq M$,
  5. go to step 1, and set $i = \xi(\tau)$.
- It is enough to simulate the paths of the chain once.
- A new simulation is needed only when the generator Q is altered.

8. Pricing synthetic CDOs

- We have now all the ingredients to price a CDO.
- The **PV01** of a CDO can be seen as a portfolio of puts $P_t(K)$ on L_t.

$$PV01 = \sum_{j=1}^{M} \Delta_i E\left[\exp\left(-\int_0^t r(\xi_u)du\right)\Phi(L_{T_j})\right], \tag{21}$$

where

$$\Phi(x) = \frac{1}{L^+ - L^-}\left[\left(L^+ - x\right)^+ - \left(L^- - x\right)^+\right], \tag{22}$$

- define...

$$P_t(K) \quad = \quad E\left[B_t^{-1}(K - L_t)^+\right] \tag{23}$$

$$\tag{24}$$

- Instead of computing the Laplace transform of default distribution, we can calculate the transform of $P_t(K)$ directly, which saves us a (time consuming) numerical integration step

$$\begin{aligned}
\hat{P}_t(\alpha) &\equiv \int_0^\infty e^{-\alpha x} P_t(x)dx \\
&= \int_{L_t}^\infty e^{-\alpha x} E\left[B_T^{-1}(x - L_t)\right]dx \\
&= \frac{1}{\alpha^2} E\exp(-\int_0^t r(\xi_u)du - \alpha L_t).
\end{aligned}$$

- The **default leg** equals the expected present value of the tranche's losses

$$DL = E\left[\int_0^T B_u^{-1} d\Xi(L_u)\right], \tag{25}$$

where $\Xi(x) = 1 - \Phi(x)$.

- Integrating by parts we can simplify (25) to

$$\begin{aligned} DL = 1 \quad &- \quad E\left[B_T^{-1}\Phi(L_T)\right] \\ &- \quad E\left[\int_0^T r(\xi_u) B_u^{-1}\Phi(L_u) du\right] \end{aligned}$$

...which can be computed using the results of the previous section.

9. A note on calibration

- How do we choose the functions $\lambda^i(\cdot)$, matrices w^i and the infinitesimal generator Q?
- ...**the market does it for us!!**
- λ^i, w^i and Q are used to match volatility CDS quotes and index quotes as well as tranche spreads.
- The number of state of the chain can be adjusted to take into account the availability of market data.

10. Example: CDX tranche spread calibration

Table 1: Market and model spreads - November 1st

	Market			Model		
	5Y	7Y	10Y	5Y	7Y	10Y
CDX	35	45	57	36.5	46.23	56.39
0 – 3%	2438	4044	5125	2438.4	4008.9	5125
3 – 7%	90	209	471	86	222.4	470.8
7 – 10%	19	46	112	19.1	45.8	99.7
10 – 15%	7	20	53	7	20.4	53.2
15 – 30%	3.5	5.75	14	3.5	5.0	14.0
30 – 100%	1.73	3.12	4	1.7	2.6	3.8

Table 2: Calibration error - November 1st

	Index	Traches
Absolute Error	1.11bp	3.77bp
Percentage Error	2.70%	3.47%

Table 3: Market and model spreads - November 2nd

	Market			Model		
	5Y	7Y	10Y	5Y	7Y	10Y
CDX	34	44	56	34.51	44	53.94
$0-3\%$	2325	3938	5056	2325	3906	5056
$3-7\%$	85.5	200	460	84.6	216.8	460
$7-10\%$	18	45.5	107	18	45.5	101
$10-15\%$	6.5	19.5	50.5	6.5	19	52.2
$15-30\%$	3.25	5.25	13.5	3.3	5.3	13.5
$30-100\%$	1.67	3.04	3.64	1.7	2.4	3.6

Table 4: Calibration error - November 2nd

	Index	Traches
Absolute Error	0.86bp	3.26bp
Percentage Error	1.73%	2.68%

Table 5: Market and model spreads - November 3th

	Market			Model		
	5Y	7Y	10Y	5Y	7Y	10Y
CDX	34	44	56	34.6	44.02	53.93
$0 - 3\%$	2325	3931	5038	2325	3892.7	5038.5
$3 - 7\%$	84.5	200	458.5	84.5	215.7	458
$7 - 10\%$	18.5	45.00	107.5	18.4	45	98.7
$10 - 15\%$	6.5	19.5	51	6.5	19.1	51.2
$15 - 30\%$	3.25	5.25	13.5	3.2	5.2	13.5
$30 - 100\%$	1.61	3.06	3.76	1.6	2.4	3.8

Table 6: Calibration error - November 3th

	Index	Traches
Absolute Error	0.90bp	3.63bp
Percentage Error	1.84%	2.55%

Table 7: Market and model spreads - November 6th

	Market			Model		
	5Y	7Y	10Y	5Y	7Y	10Y
CDX	33	43	54	34.61	43.88	53.97
0 – 3%	2256	3863	4963	2255.9	3794.3	4963.1
3 – 7%	77	192	438	77	201.3	438
7 – 10%	17	41	98	17	41	93.5
10 – 15%	6	18.5	46.5	6	17.1	47
15 – 30%	3.13	5.75	12	3.1	5.2	12.8
30 – 100%	1.27	2.55	3.23	1.3	2	3.2

Table 8: Calibration error - November 6th

	Index	Traches
Absolute Error	0.84bp	4.81bp
Percentage Error	2.33%	3.44%

Figure 1: Calibrated portfolio loss density. 5Y Maturity. X axe: L_t, Y axe: density

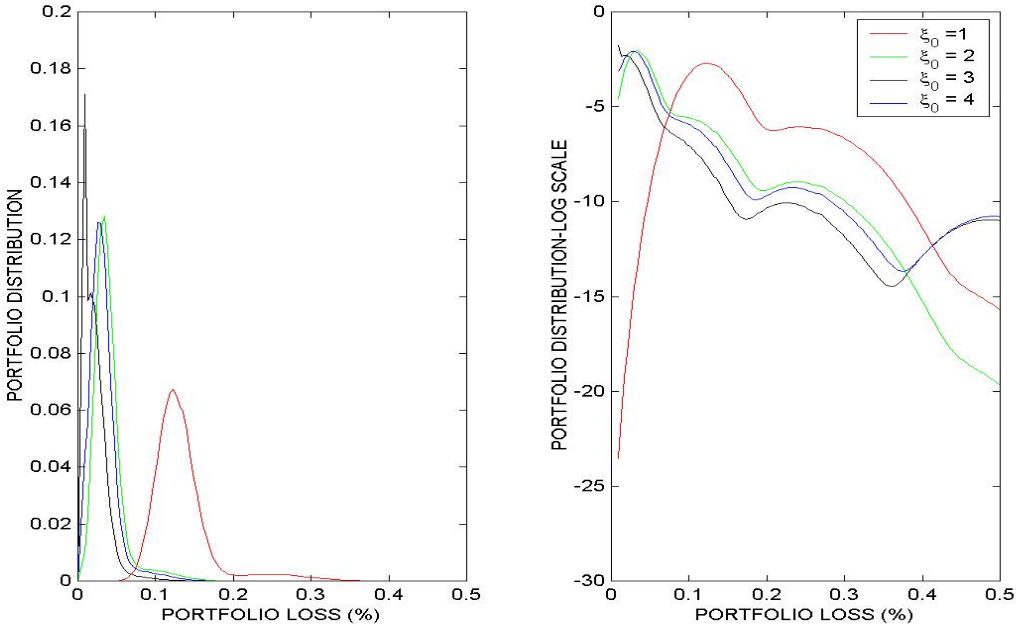


Figure 2: Calibrated portfolio loss density. 7Y Maturity. X axe: L_t, Y axe: density

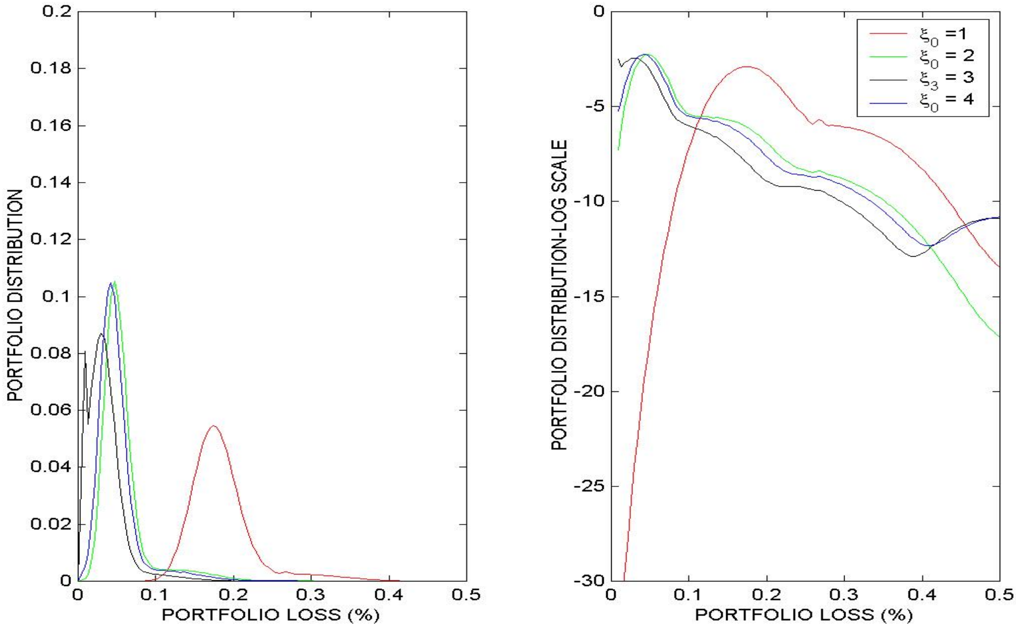
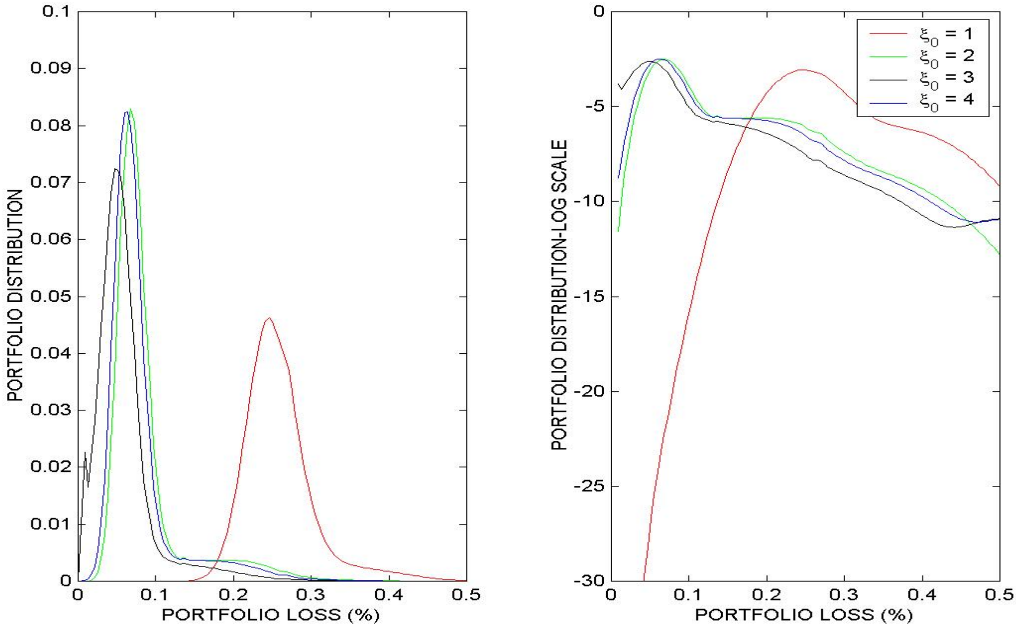


Figure 3: Calibrated portfolio loss density. 10Y Maturity. X axe: L_t, Y axe: density



11. What's next?... Modeling the *defaultable* equity market

- Stock prices can be seen as the expected sum of all future dividends, up to default, appropriately discounted.

- We assume that the continuous stochastic dividend paid by the firm is given by

$$\frac{d\delta_t}{\delta_t} = \mu(\xi_t)dt + \sigma(\xi_t)dW_t. \tag{26}$$

- Recall that the conditional probability of survival is given by

$$q_t \equiv P\left(\tau > t \mid \mathcal{F}_t^{\xi}\right) = \exp\left(-\int_0^t \lambda(\xi_u)du\right), \tag{27}$$

- The stock price at t can be derived as follows

$$S_t \equiv E_t\left[\int_t^{\tau} B_u^{-1}\delta_u du\right] \tag{28}$$

$$= \delta_t v(\xi_t). \tag{29}$$

where $\tilde{\mu} \equiv \mu - \lambda - r$ and

$$v(\xi) \equiv -(\tilde{\mu} + Q)^{-1}\mathbf{1}(\xi). \tag{30}$$

12. Single name equity-credit hybrid options

- The price of a *no-default put* option with maturity T is given by

$$P_T(k) \equiv E\left[B_T^{-1}\left(e^k - e^s\right)^+; \tau \geq T\right]. \tag{31}$$

  where $k \equiv \log(K)$ and $s \equiv \log(S_T)$.

- The Laplace transform (w.r.t the log-strike) of the *no-default put* can be computed explicitly

$$\begin{aligned}
\widehat{P}_T(\eta) &\equiv \int_{-\infty}^{\infty} e^{-\eta k} P_T(k) dk \\
&= \int_{-\infty}^{\infty} e^{-\eta k} E\left[B_T^{-1}\left(e^k - e^s\right)^+; \tau \geq T\right] dk \\
&= \frac{\delta_0^{1-\eta}}{\eta(\eta-1)} e^{(Q+z_\eta)T} \hat{v}(\xi_0)
\end{aligned}$$

  where $\hat{v}(\xi) \equiv v(\xi)^{1-\eta}$ for all $\xi \in \{1, \ldots, N\}$ and

$$z_\eta \equiv (1-\eta)(\mu - \frac{1}{2}\sigma^2) + \frac{1}{2}(1-\eta)^2\sigma^2 - r - \lambda. \tag{32}$$

- Remark: The transform of the classical vanilla put can be recovered from the previous calculation simply by setting the vector $\lambda = 0$.