ABOUT THE PRICING EQUATION IN FINANCE

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We derive the pricing equation of a general (American or Game) Contingent Claim in the set-up of a rather generic Markovian factor process.

As an aside we establish the convergence of *stable, monotone and consistent* approximation schemes to the pricing function (solution to the pricing equation).

1 Model

We first introduce a flexible Markovian model $\mathcal{X} = (X, N)$ made of a jump diffusion X interacting with a pure jump process N (which in the simplest case reduces to a Markov chain in continuous time).

The state space is $\mathcal{E} = [0, T] \times \mathbb{R}^d \times I$ with $I = \{1, \cdots, k\}$, where $T > 0$ is a finite horizon and d and k denote positive integers.
Note that a function $u = u(t, x, i)$ on $[0, T] \times \mathbb{R}^d \times I$ may equivalently be considered as a *system* $u = (u^i)_{i \in I}$ of functions $u^i = u^i(t, x)$ on $[0, T] \times \mathbb{R}^d$. Likewise we denote $u(t, x, i, j)$, or $u^{i,j}(t, x)$ for a function u of (t, x, i, j).

Let us be given a stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$ on $[0, T]$ endowed with a d-dimensional Brownian motion B and an *integer-valued random measure χ*. Our model consists of an $\mathbb{R}^d \times I$-valued Markov càdlàg process $\mathcal{X} = (X, N)$ on $[0, T]$ with initial condition (x, i) at time 0, such that the $\mathbb{R}^d$-valued process

X satisfies:

$$dX_t = b(t, X_t, N_t)\, dt + \sigma(t, X_t, N_t)\, dB_t + \int_{\mathbb{R}^d} \delta(t, X_{t-}, N_{t-}, y)\, \widetilde{\chi}(dy, dt)$$

where:

- the compensated jump measure $\widetilde{\chi}$ of χ is given by

$$\widetilde{\chi}(dt, dy) = \chi(dt, dy) - f(t, X_t, N_t, y)m(dy)dt\ ,$$

so that the *intensity measure of the jumps of X* is given by $f(t, X_t, N_t, y)$, $y \in \mathbb{R}^d$;

- the *intensity measure of the I-valued pure jump process N* is given by $\mathbb{1}_{\{N_t \neq j\}}\lambda(t, X_t, N_t, j)$, $j \in I$.

In these equations $m(dy)$ is a *jump probability measure* on $\mathbb{R}^d \setminus \{0\}$, and all the coefficients are Borel–measurable functions such that:

- the $\sigma^i(t, x)$ are d-dimensional *dispersion* matrices, with related *covariance*

matrices $a^i(t,x) = \sigma^i(t,x)\sigma^i(t,x)^\mathsf{T}$;

● the $b^i(t,x)$ are d-dimensional *drift* vector coefficients;

● the *intensity* functions $f^i(t,x,y)$ are bounded, and the *jump size* functions $\delta^i(t,x,y)$ are bounded w.r.t y at fixed (t,x), locally uniformly in (t,x);

● the $\left[\lambda^{i,j}(t,x)\right]_{i,j\in I}$ are *intensity* matrices such that the functions $\lambda^{i,j}(t,x)$ are non-negative and bounded for $i \neq j$, and

$\lambda^{i,i}(t,x) = -\sum_{j\in I\setminus\{i\}} \lambda^{i,j}(t,x)$.

This model admits versatile applications in financial modeling.

2 Main Result

Let us define the following linear operator $\mathcal{G}$ acting on regular functions $u = u^i(t, x)$ for $(t, x, i) \in \mathcal{E}$, and where ∂u (resp. $\mathcal{H}u$) denotes the row-gradient (resp. Hessian) of $u(t, x, i) = u^i(t, x)$ with respect to x :

$$\mathcal{G}u^i(t, x) = \partial_t u^i(t, x) + \tfrac{1}{2}\mathrm{Tr}[a^i(t, x)\mathcal{H}u^i(t, x)]$$

$$+\partial u^i(t, x) \left(b^i(t, x) - \int_{\mathbb{R}^d} \delta^i(t, x, y) f^i(t, x, y) m(dy) \right) + \mathcal{I}u^i(t, x)$$

where

$$\mathcal{I}u^i(t, x) = \int_{\mathbb{R}^d} \left(u^i(t, x + \delta^i(t, x, y)) - u^i(t, x) \right) f^i(t, x, y) m(dy) .$$

Let us further be given a system of real-valued continuous *cost functions* with polynomial growth in x, namely a *running cost function* $g^i(t, x, u, z, r)$ (where $(u, z, r) \in \mathbb{R}^k \times \mathbb{R}^{1 \otimes d} \times \mathbb{R}$), a *terminal cost function* $\Psi^i(x)$, and *lower and upper cost functions* $\ell^i(t, x)$ and $h^i(t, x)$, such that:

- g is Lipschitz continuous with respect to (u, z, r) and non-decreasing with respect to r ;
- $\ell \leq h$ and $\ell(T, \cdot) \leq \Psi \leq h(T, \cdot)$;
- $\ell = \phi \vee c$ for a 'regular' function ϕ on $\mathcal{E}$ and for a constant $c \in \mathbb{R} \cup \{-\infty\}$.

We denote by $(\mathcal{V}2)$ the following *variational inequality with double obstacle problem* (*reaction-diffusion system*):

$$
\max\Big(\min\Big(\quad -\mathcal{G}u^i(t,x) - g^i(t,x,u(t,x),(\partial u \sigma)^i(t,x), \mathcal{I}u^i(t,x)),
$$
$$
u^i(t,x) - \ell^i(t,x)\Big), u^i(t,x) - h^i(t,x)\Big) = 0
$$

on $\{t < T\}$ supplemented by the terminal condition Ψ (the terminal cost function) at T.

Note that as opposed to the set-up of Becherer and Schweizer [7] where *linear* reaction-diffusion systems of parabolic equations are considered in a diffusion model with regimes (but *without jumps* in X), here, due to the presence of the *obstacles* in the problem and of the *jumps* in X, the related reaction-diffusion

system $(\mathcal{V}2)$ typically does not have *classic* solutions.

By a *solution* to $(\mathcal{V}2)$, we thus mean a *viscosity solution* with polynomial growth in x to $(\mathcal{V}2)$, adapting the general definitions of viscosity solutions for nonlinear PDEs (see [14, 19]) to (finite activity) *jumps* and *systems* of PIDEs as in [1, 22, 4, 13]. A complementary *weak Sobolev solutions approach* is dealt with in Crépey–Matoussi [17].

The notions of *stable, monotone and consistent approximation schemes* below were originally introduced for nonlinear PDEs by Barles and Souganidis [6], and further extended to PIDEs by Briani, La Chioma and Natalini [13]).

Theorem 2.1 *'Under suitable conditions:'*
(i) *$(\mathcal{V}2)$ admits a unique solution u. Moreover, u admits a probabilistic interpretation (Feynman–Kac formula) in terms of a related control problem (doubly reflected BSDE).*
(ii) *Any stable, monotone and consistent approximation scheme (u_h) converges locally uniformly to u as $h \to 0$.*

Remark 2.1 **(i)** In part (i), existence and representation are obtained by BSDE techniques (see Crépey and Matoussi [18], Crépey [16]) whereas uniqueness results from typical viscosity arguments (cf. in particular Barles et al. [4]).
(ii) Part (ii) extends to models with regimes (whence *systems* of PIDEs) the convergence results of [6, 13].

3 Application in Finance

We now specify the above set-up to risk-neutral pricing problems in finance. In this case:

- the function g is typically of the form:

$$g^i(t, x, u, z, r) = c^i(t, x) - \mu^i(t, x)u^i + \sum_{j \in I \setminus \{i\}} (u^j - u^i)\lambda^{i,j}(t, x) , \quad (1)$$

for dividend and interest-rate related functions c and μ (or dividends and interest-rates *adjusted for credit spread*, in a more general context of *defaultable contingent claims*);

- $\Psi(\mathcal{X}_T)$ corresponds to a *terminal payoff* that is paid by the issuer to the holder at time T if the contract was not exercised before T;
- $\ell(t, \mathcal{X}_t)$, resp. $h(t, \mathcal{X}_t)$, corresponds to a *lower*, resp. *upper payoff* that is paid by the issuer to the holder of the claim in the event of early termination of the contract at the initiative of the holder, resp. issuer.

The contingent claims under consideration are thus general *Game Contingent*

Claims (see [8, 9, 10, 11]), covering American Claims (and European Claims) as special cases. From the point of view of the financial interpretation, the components of $\mathcal{X}$ are observable *factors*.

Along with this factor process $\mathcal{X}$, we consider a *primary market* composed of the savings account and of $d + k$ primary risky assets with locally bounded semimartingale price process $\mathcal{Y} = \left(Y^1, \cdots, Y^{d+k}\right)^{\mathsf{T}}$. The discount factor β, that is, the inverse of the savings account, is defined as $\beta_t = \exp(-\int_0^t \mu_u \, du)$ (so μ can be interpreted as a *short-term interest rate* process). For simplicity of presentation we assume no dividends on the primary risky assets.
Consistently with arbitrage requirements on the underlying market, we postulate that $\beta \mathcal{Y}$ is a $\mathbb{P}$ – local martingale, where the probability measure $\mathbb{P}$ is equivalent to the objective probability measure on the underlying market.

The *factor process* $\mathcal{X}$ and the *primary price process* $\mathcal{Y}$ are typically intimately, though non-trivially, connected, as follows:
● Most factors are typically given as primary price processes. The components of $\mathcal{X}$ that are not included in $\mathcal{Y}$ (if any) are to be understood as simple factors that

may be required to 'Markovianize' the payoffs of a game option (e.g., factors accounting for path dependence in the option's payoff and/or non-traded factors such as stochastic volatility in the dynamics of the assets underlying the option);

● Some of the primary price processes may not be needed as factors, but are used for hedging, so that condition (2) may be satisfied below.

Theorem 3.1 **(i) (Arbitrage)** *The process $u(t, X_t, N_t)$, $t \in [0, T]$ is an arbitrage price process for the Claim.*

(ii) (Hedging; *Work in progress*) *$u(0, \mathcal{X}_0)$ is the minimal initial wealth of an (issuer super-)hedge with $\mathbb{P}$ – local martingale residual cost process for the Claim. Denoting by $\widetilde{\nu}$ the compensated jump measure of N, let us assume further that*

$$\beta_t d \begin{pmatrix} B_t \\ \widetilde{\nu}_t \end{pmatrix} = \Lambda_t \, d \left(\beta_t \mathcal{Y}_t \right), \quad t \in [0, T], \tag{2}$$

for some $\mathbb{R}^{d+k \otimes d+k}$-valued predictable locally bounded process Λ. Then a

hedge with $\mathbb{P} - local$ *martingale residual cost process*

$$dn_t = \int_{\mathbb{R}^d} \left[u\left(t, X_{t-} + \delta(t, X_{t-}, N_{t-}, y), N_{t-}\right) - u(t, X_{t-}, N_{t-}) \right] \widetilde{\chi}(dy, dt)$$

is given by

$$\zeta_t = \left[\partial u \sigma(t, X_t, N_t), u(t, X_{t-}) - u(t, X_{t-}, N_{t-}) \right] \Lambda_t, \quad t \in [0, T]$$

$$\tau = \inf \left\{ t \in [0, T] \, ; \, u(t, X_t, N_t) \in \mathcal{C} \right\} \wedge T$$

where

$$u(t, x) - u^i(t, x) := \left(u^1(t, x) - u^i(t, x), \cdots, u^k(t, x) - u^i(t, x) \right),$$

and where

$$\mathcal{C} := \left\{ (t, x, i) \in \mathcal{E} \, ; \, u^i(t, x) = h^i(t, x) \right\}$$

is the Early Call Region*.*

Remark 3.1 (i) In the special case of a purely diffusive (no jumps nor regimes),

complete market, with morally $\Lambda = \sigma^{-1}$, we recover the usual relation $\zeta_t = \partial u(t, X_t)$, ensuring a self-financing (super-)hedge to the issuer of the claim.

(ii) These results can be extended to defaultable Contingent Claims by passage to an equivalent fictitious default-free world with suitably modified discount factors and dividend processes (introducing a further defaultable primary asset to hedge default risk, if wished).

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