# Hedging and Optimization in a Geometric Additive Market.

José M. Corcuera

Faculty of Mathematics
University of Barcelona

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# Outline

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1. Hedging in an additive model
   - The Market model
   - The stock price formula
   - Equivalent Martingale Measures
   - Power-Jump Processes
   - Example
2. Portfolio optimization
   - Utility functions
   - Optimal wealth
   - Example
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3. References

- Our market model, denoted by $M$, will be a (stochastic) exponential additive model consisting of a riskfree bond $B = \{B_t, t \geq 0\}$, where $B_t = \exp(\int_0^t r_s \mathrm{d}s)$, with $r_s$ deterministic, and a risky stock $S = \{S_t, t \geq 0\}$ which verifies

$$\frac{\mathrm{d}S_t}{S_{t-}} = \mathrm{d}Z_t, \qquad S_0 > 0, \tag{1}$$

where $Z$ is a natural additive process with local characteristics (with respect to the Lebesgue measure) $(c_t^2, \nu_t, \gamma_t)$.

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$$\frac{\mathrm{d}S_t}{S_{t-}} = \mathrm{d}Z_t, \qquad S_0 > 0, \tag{1}$$

where $Z$ is a natural additive process with local characteristics (with respect to the Lebesgue measure) $\left(c_t^2, \nu_t, \gamma_t\right)$.

- Except when $(Z_t)$ is a Brownian motion or a Poisson process, the above described models are incomplete: contingent claims cannot, in general, be hedged by a self-financing portfolio. This is equivalent to the fact that there are many equivalent "martingale measures": probability measures (equivalent to the original one) under which the discounted stock values are martingales.

From the Lévy-Itô decomposition, one can assume that $Z$ in (1) can be written as

$$Z_t = \int_0^t c_s \mathrm{d} W_s + X_t, \tag{2}$$

where $W = \{W_t, t \in [0, T]\}$ is a standard Brownian motion and $X = \{X_t, t \in [0, T]\}$ is a jump process independent of $W$. Moreover, the jump part is given by

$$X_t = \int_{\{s \in (0,t], |x|<1\}} x \left(J(\mathrm{d}s, \mathrm{d}x) - \nu_s(\mathrm{d}x)\mathrm{d}s\right) \tag{3}$$

$$+ \int_{\{s \in (0,t], |x| \geq 1\}} x J(\mathrm{d}s, \mathrm{d}x) + \int_0^t \gamma_s \mathrm{d}s, \tag{4}$$

where $J(\mathrm{d}t, \mathrm{d}x)$ is a Poisson random measure on $[0, T] \times \mathbb{R} \backslash \{0\}$ with intensity measure $\nu_t\,(\mathrm{d}x)\,\mathrm{d}t$

We assume that the family of Lévy measures $\{\nu_t\}_{t\in[0,T]}$ satisfies, for some $\varepsilon > 0$ and $\lambda > 0$,

$$\sup_{t\in[0,T]} \int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda|x|)\nu_t(\mathrm{d}x) < \infty. \tag{5}$$

As a consequence of this assumption, it is easy to show that

$$\int_{-\infty}^{+\infty} |x|^i \, \nu_t(\mathrm{d}x) < \infty$$

for all $i \geq 2$ and all $t \in [0, T]$.

Moreover, with these assumptions, the Doob decomposition of $X$, in terms of a martingale part and a predictable process of finite variation, is given by

$$X_t = L_t + \int_0^t a_s \mathrm{d}s,$$

where $L = \{L_t, t \geq 0\}$ is a martingale and $E_P[X_t] = \int_0^t a_s \mathrm{d}s$.

If we denote $M(\mathrm{d}t,\mathrm{d}x) = J(\mathrm{d}t,\mathrm{d}x) - \mathrm{d}t\nu_t(\mathrm{d}x)$ the martingale part of $X$ can be written in terms of the compensated Poisson random measure $M(\mathrm{d}t,\mathrm{d}x)$ as

$$L_t = \int_0^t \int_{-\infty}^{+\infty} xM(\mathrm{d}s, \mathrm{d}x).$$

So, in our case the Lévy-Itô decomposition is

$$Z_t = \int_0^t c_s \mathrm{d}W_s + \int_0^t \int_{-\infty}^{+\infty} x M(\mathrm{d}s, \mathrm{d}x) + \int_0^t a_s \mathrm{d}s$$

Using Itô's formula for semimartingales one can show that Equation (1) has the solution

$$S_t = S_0 \exp\left(Z_t - \frac{1}{2}\int_0^t c_s^2 \mathrm{d}s\right) \prod_{0<s\le t} (1+\Delta Z_s)\exp(-\Delta Z_s). \qquad (6)$$

In order to ensure that $S_t > 0$ for all $t \geq 0$ *a.s.*, we require that $\Delta Z_t > -1$ for all $t$. Hence, we shall assume that the family of Lévy measures $\{\nu_t\}_{t\in[0,T]}$ is supported on $(-1,+\infty)$.
It is interesting to note that we can also write:

$$S_t = S_0 \exp(\bar{Z}_t),$$

where

$$\begin{aligned}\bar{Z}_t &= \int_0^t c_s \mathrm{d}W_s + \int_0^t \int_{-\infty}^{+\infty} \log(1+x) M(\mathrm{d}s, \mathrm{d}x) \\ &\quad + \int_0^t (a_s - \frac{c_s^2}{2})\mathrm{d}s + \int_0^t \int_{-\infty}^{+\infty} (\log(1+x) - x)\nu_s(\mathrm{d}x)\mathrm{d}s.\end{aligned}$$

So, stochastic exponential models are the same as usual exponential models. They are simply two ways of expressing the same model.

We look for *structure preserving*, $P$-equivalent, martingale measures $Q$. Under these probabilities $Z$ remains an additive process, the process $\tilde{S} = \{\tilde{S}_t = \exp(-\int_0^t r_s \mathrm{d}s) S_t,\, 0 \leq t \leq T\}$ is a $Q$-martingale and $Q$ and $P$ are equivalent.

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We have the following results,

## Theorem

*Let $Z = \{Z_t, 0 \leq t \leq T\}$ be an additive process with local characteristics $(c_t^2, \nu_t, \gamma_t)$. Then, if there is a probability measure Q equivalent to P, such that Z is a Q-(natural) additive process with local characteristics (with respect to the Lebesgue measure) $(\bar{c}_t^2, \bar{\nu}_t, \bar{\gamma}_t)$ we have:*

(i) $\bar{\nu}_t(dx) = H(t,x)\nu_t(dx)$ *for some Borel function* $H(t,x) : \mathbb{R}^+ \times \mathbb{R} \to (0,\infty)$.

(ii) $\bar{\gamma}_t = \gamma_t + \int_{-\infty}^{+\infty} x\mathbf{1}_{\{|x|<1\}}(H(t,x)-1)\nu_t(dx) + G_t c_t^2$ *for some Borel function* $G_t : \mathbb{R}^+ \to (0,\infty)$.

(iii) $\bar{c}_t = c_t$.

*for every* $0 \leq t \leq T$.

## Theorem

*Suppose that we are in the conditions of the previous theorem, then the density process* $\{\frac{\mathrm{d}Q_t}{\mathrm{d}P_t} = \xi_t, 0 \leq t \leq T\}$ *is given by*

$$\xi_t = \exp\Bigg(\int_0^t G_s c_s \mathrm{d}W_s - \frac{1}{2}\int_0^t G_s^2 c_s^2 \mathrm{d}s + \lim_{\varepsilon\to 0}\left(\int_0^t \int_{|x|>\varepsilon} \log H(s,x) J(\mathrm{d}t,\mathrm{d}x) - \int_0^t \int_{|x|>\varepsilon} (H(s,x)-1)\nu_s(\mathrm{d}x)\mathrm{d}s\right)\Bigg) \tag{7}$$

*with* $E_P[\xi_t] = 1$, *for every* $t \in [0,T]$*. The convergence on the right-hand side of (7) is uniform in t on any bounded interval, P-a.s.*

The previous theorems imply that the process $\bar{W} = \{\bar{W}_t, 0 \leq t \leq T\}$ with

$$\bar{W}_t = W_t - \int_0^t G_s c_s \mathrm{d}s$$

is a Brownian motion under $Q$ and also, if $\nu_t$ and $\bar{\nu}_t$ verify the moment-condition (5), the process $X$ is a jump additive process process with $Q$-Doob-Meyer decomposition

$$X_t = \bar{L}_t + \int_0^t a_s \mathrm{d}s + \int_0^t \int_{-\infty}^{+\infty} x(H(s,x) - 1)\nu(\mathrm{d}x)\mathrm{d}s,$$

where $\bar{L} = \{\bar{L}_t, 0 \leq t \leq T\}$ is a $Q$-martingale and where $\bar{\nu}_t(\mathrm{d}x) = H(t,x)\nu_t(\mathrm{d}x) \quad \forall 0 \leq t \leq T$.

Now, we want to find an equivalent martingale measure $Q$ under which the discounted price process $\tilde{S}$ is a martingale. Observing that $\Delta L_t = \Delta \bar{L}_t$, we have from (6) that

$$\tilde{S}_t = S_0 \exp\left(\int_0^t c_s \mathrm{d}\bar{W}_s + \bar{L}_t + \int_0^t \left(a_s - r_s + G_s c_s^2 - \frac{c_s^2}{2}\right) \mathrm{d}s\right)$$
$$\times \exp\left(\int_0^t \int_{-\infty}^{+\infty} x(H(s,x)-1)\nu_s(\mathrm{d}x)\right) \mathrm{d}s \prod_{0<s\leq t} (1+\Delta\bar{L}_s)\exp(-\Delta\bar{L}_s).$$

Then a necessary and sufficient condition for $\tilde{S}$ to be a $Q$-martingale is $G_t$ and $H(t,x)$ to verify

$$G_t c_t^2 + a_t - r_t + \int_{-\infty}^{+\infty} x(H(t,x)-1)\nu_t(\mathrm{d}x) = 0.$$

$0 \leq t \leq T$. Note that,

$$Z_t = \int_0^t c_s \mathrm{d}\bar{W}_s + \bar{L}_t + \int_0^t r_s \mathrm{d}s$$

The following transformations of $Z = \{Z_t, t \geq 0\}$ will play an important role in our analysis. We set

$$Z_t^{(i)} = \sum_{0<s\leq t} (\Delta Z_s)^i, \qquad i \geq 2,$$

where $\Delta Z_s = Z_s - Z_{s-}$. Define the $Q$-martingales

$$H_t^{(i)} = Z_t^{(i)} - E_Q(Z_t^{(i)}), \qquad i = 1, 2, \dots,$$

with $Z_t^{(1)} = Z_t$. We have the following result

## Theorem (Nualart-Schoutens-Balland)

*Any $Q$-square-integrable martingale $M_t$ can be expressed as*

$$M_t = M_0 + \sum_{k=1}^{\infty} \int_0^t \beta_s^n \mathrm{d}\bar{H}_s^{(n)}$$

*where $\bar{H}_s^{(n)}$ are the orthogonal version of the $H^{(n)}$ defined previously and the $\beta^i$ are predictable processes.*

Following Corcuera *et al.*(2005), we complete our market, $M$, with a series of additional assets, $Y^{(i)} = \{Y_t^{(i)}, t \geq 0\}$, based on the above mentioned processes:

$$Y_t^{(i)} = e^{\int_0^t r_s \mathrm{d}s} H_t^{(i)}, \qquad i \geq 2.$$

We shall call them "power-jump" assets.

## Theorem

*The market model, $M_Q$, obtained by enlarging the market $M$ with the power-jump assets is complete, in the sense that any square-integrable contingent claim $X \in L^2(Q)$ can be replicated by an (admissible) self-financing portfolio.*

Let $X$ be a square-integrable (with respect to $Q$) contingent claim. Consider the squared-integrable martingale $M_t := E(e^{-\int_0^T r_s \mathrm{d}s} X | \mathcal{F}_t)$. By the previous theorem we can write

$$\begin{aligned} \mathrm{d}M_t &= \sum_{k=1}^{\infty} \beta_t^n \mathrm{d}\bar{H}_t^{(n)} \\ &= \beta_t^1 \frac{\mathrm{d}\tilde{S}_t}{\tilde{S}_{t-}} + \sum_{k=2}^{\infty} \beta_t^k \mathrm{d}\,\tilde{Y}_t^{(k)} \end{aligned}$$

Then if we take a self-financing portfolio: $((\phi_t^i)_{i\geq 1})_{0\leq t\leq T}$, where $\phi^1$ denotes the number of units of the stock, and $(\phi^i)_{i\geq 2}$ the number of jump-power assets of different order, we will have that the discounted value of this portfolio evolves as

$$\mathrm{d}\tilde{V}_t = \phi_t^1 \mathrm{d}\tilde{S}_t + \sum_{k=2}^{\infty} \phi_t^n \mathrm{d}\tilde{Y}_t^{(n)}.$$

So, by taking $\phi_t^1 = \frac{\beta_t^1}{\tilde{S}_{t-}}$ and $\phi_t^i = \beta_t^i$ we obtain the replicating portfolio.

In certain cases we can obtain hedging formulas directly, by using the Itô formula. In fact assume that the discounted price of the option at time $t$ can be written as $\tilde{F}(s, S_s)$, $F$ smooth, then by the Itô formula

$$\begin{aligned} & \mathrm{d}\tilde{F}(s, S_s) \\ = \;& \frac{\partial F}{\partial S_s}\mathrm{d}\tilde{S}_s \\ & + \int_{-\infty}^{+\infty} \left( \tilde{F}(s, S_{s-}(1+y)) - \tilde{F}(s, S_{s-}) - y\tilde{S}_{s-}\frac{\partial \tilde{F}}{\partial \tilde{S}_s} \right) \bar{M}(\mathrm{d}s, \mathrm{d}y) \end{aligned}$$

where $\bar{M}(\mathrm{d}t, \mathrm{d}y) = J(\mathrm{d}t, \mathrm{d}y) - \mathrm{d}t\bar{\nu}_t(\mathrm{d}y)$

Then if we assume now that $\tilde{F}(s, S_{s-}(1+y))$ can be expanded as a series of powers in $y$ we have

$$
\begin{aligned}
& \mathrm{d}\tilde{F}(s, S_s) \\
= \;& \frac{\partial F}{\partial S_s}\mathrm{d}\tilde{S}_s + \int_{-\infty}^{+\infty} \sum_{k\geq 2} \frac{1}{k!} \frac{\partial^k \tilde{F}}{\partial y^k}_{|y=0} y^k \bar{M}(\mathrm{d}s, \mathrm{d}y) \\
= \;& \frac{\partial F}{\partial S_s}\mathrm{d}\tilde{S}_s + \sum_{k\geq 2} \frac{1}{k!} \frac{\partial^k \tilde{F}}{\partial y^k}_{|y=0} \mathrm{d}\,\tilde{Y}_s^{(k)}
\end{aligned}
$$

For instance, consider derivatives with payoff $S_T^k$, $k \geq 2$. Then its discounted price will be given by

$$\tilde{F}^{(k)}(t, S_t) = e^{-\int_0^T r_s \mathrm{d}s} E_Q(S_T^k | \mathcal{F}_t) = e^{-\int_0^T r_s \mathrm{d}s} S_t^k E_Q\left(\left(\frac{S_T}{S_t}\right)^k | \mathcal{F}_t\right)$$

$$= e^{-\int_0^T r_s \mathrm{d}s} S_t^k E_Q\left(\left(\frac{S_T}{S_t}\right)^k\right) = \varphi^{(k)}(t, T) S_t^k$$

Then this derivative can be replicated by using the power-jump assets

$$\mathrm{d}\tilde{F}^{(k)}(t, S_t) = \frac{kF^{(k)}(t, S_{t-})}{S_{t-}}\mathrm{d}\tilde{S}_t + \sum_{i=2}^{k} \tilde{F}^{(k)}(t, S_{t-})\binom{k}{i}\mathrm{d}\,\tilde{Y}_t^{(i)}.$$

Define $\tilde{F}^{(1)}(t, S_t) = \tilde{S}_t$, and since:

$$\mathrm{d}\tilde{Y}_t^{(1)} = \frac{\mathrm{d}\tilde{S}_t}{\tilde{S}_{t-}},$$

we can write

$$\mathrm{d}\tilde{F}^{(k)}(t, S_t) = \sum_{i=1}^{k} e^{-\int_0^t r_s \mathrm{d}s} F^{(k)}(t, S_{t-}) \binom{k}{i} \mathrm{d}\tilde{Y}_t^{(i)}$$

and

$$\mathrm{d}\tilde{Y}_t^{(k)} = \sum_{i=1}^{k} \binom{k}{i} (-1)^{k-i} \frac{1}{\tilde{F}^{(i)}(t, S_{t-})} \mathrm{d}\tilde{F}^{(i)}(t, S_t).$$

Moreover if we want to hedge in terms of options we can use the equality:

$$E_Q(e^{-\int_0^T r_s \mathrm{d}s} f(S_T)|\mathcal{F}_t) = e^{-\int_0^T r_s \mathrm{d}s} f(0) + f'(0)\tilde{S}_t + \int_0^\infty f''(K)\tilde{C}_t(K)\mathrm{d}K$$

where $\tilde{C}_t(K) := e^{-\int_0^T r_s \mathrm{d}s} E_Q((S_T - K)_+|\mathcal{F}_t)$ and $f$ is any smooth function. This formula provides a static hedge of the payoff $f(S_T)$. Then, for $k \geq 2$,

$$\begin{aligned} &\mathrm{d}\tilde{F}^{(k)}(t, S_t) \\ = &\int_0^\infty k(k-1)K^{k-2}\mathrm{d}\tilde{C}_t(K)\mathrm{d}K \end{aligned}$$

**Theorem**

*The market $M$, enlarged with call options with the same maturity $T$ and different strikes is a complete market.*

We know that $(Y_t^{(i)}), i \geq 1$ is a total set of assets, then form any $X \in L^2(Q)$ we have that the discounted value of the replicating portfolio, say $\tilde{V}_t$ can be written as

$$
\begin{aligned}
\mathrm{d}\tilde{V}_t &= \lim_m \sum_{k=1}^{m} \phi_s^{(k,m)} \mathrm{d}\tilde{Y}_s^{(k)} \\
&= \lim_m \sum_{k=1}^{m} \sum_{i=1}^{k} \binom{k}{i} \phi_s^{(k,m)} (-1)^{k-i} \frac{1}{\tilde{F}^{(i)}(t, S_{t-})} \mathrm{d}\tilde{F}^{(i)}(t, S_t) \\
&= \lim_m \sum_{k=1}^{m} k \phi_s^{(k,m)} (-1)^{k-1} \frac{1}{\tilde{S}_{t-}} \mathrm{d}\tilde{S}_t \\
&+ \lim_m \sum_{k=2}^{m} \sum_{i=2}^{k} \binom{k}{i} \phi_s^{(k,m)} (-1)^{k-i} \frac{\int_0^\infty K^{i-2} \mathrm{d}\tilde{C}_t(K) \mathrm{d}K}{\int_0^\infty K^{i-2} \tilde{C}_t(K) \mathrm{d}K}
\end{aligned}
$$

In some special cases this simplifies to

$$\mathrm{d}\tilde{V}_t = \frac{\mathrm{d}F}{\mathrm{d}S_t}_{|S_t=0}\mathrm{d}\tilde{S}_t + \sum_{k=2}^{\infty} \frac{\mathrm{d}^i F}{\mathrm{d}S_t^i}_{|S_t=0} \frac{\int_0^\infty K^{i-2}\mathrm{d}\tilde{C}_t(K)\mathrm{d}K}{\int_0^\infty K^{i-2}\tilde{C}_t(K)\mathrm{d}K}$$

where $F$ is the price function of the derivative.

Consider an Asian option struck at $K$, that is an option with payoff

$$X = \left( \frac{1}{T} \int_0^T S_u \mathrm{d}u - K \right)_+ .$$

in an additive market where $B_t = e^{\int_0^t r_s \mathrm{d}s}$. Then the price process is

$$G(t, S_t, V_t) = \frac{B_t}{B_T} E_Q \left[ X | \mathcal{F}_t \right],$$

where $V_t := \frac{1}{T} \int_0^t S_u \mathrm{d}u$ and $X = (V_T - K)_+$. In fact, we have

$$\begin{aligned} E_Q \left[ X | \mathcal{F}_t \right] &= E_Q \left[ \left( \frac{1}{T} \int_0^T S_u \mathrm{d}u - K \right)_+ \middle| \mathcal{F}_t \right] \\ &= S_t E_Q \left[ \left( \frac{1}{T} \int_t^T \frac{S_u}{S_t} \mathrm{d}u + x \right)_+ \right]_{x = \frac{V_t - K}{S_t}} \\ &= S_t \phi \left( t, \frac{U_t}{S_t} \right), \end{aligned}$$

where $U_t := V_t - K$ and $\phi(t,x) := E_Q\left[\left(\frac{1}{T}\int_t^T \frac{S_u}{S_t}\mathrm{d}u + x\right)_+\right]$ is a deterministic function. Hence,

$$G(t, S_t, V_t) = \frac{B_t}{B_T} S_t \phi\left(t, \frac{U_t}{S_t}\right).$$

In order to obtain this price function we can solve the PIDE

$$
\begin{aligned}
&D_0 G(t, x_1, x_2) + \frac{1}{T} x_1 D_2 G(t, x_1, x_2) + r_t x_1 D_1 G(t, x_1, x_2) \\
&+ \frac{1}{2} c_t^2 x_1^2 D_1^2 G(t, x_1, x_2) + \mathcal{D} G(t, x_1, x_2) = r_t G(t, x_1, x_2), \\
&G(T, x_1, x_2) = (x_2 - K)_+ .
\end{aligned}
$$

where

$$
\begin{aligned}
&\mathcal{D} G(t, x_1, x_2) := \\
&\int_{-\infty}^{+\infty} (F(t, x_1(1+y), x_2)) - G(t, x_1, x_2) - x_1 y D_1 G(t, x_1, x_2)) \, \bar{\nu}_t(\mathrm{d}y) .
\end{aligned}
$$

In terms of the function $\phi(t,x)$ the PIDE can be written as

$$\frac{\partial}{\partial t}\phi(t,x)+\left(\frac{1}{T}-r_t x\right)\frac{\partial}{\partial x}\phi(t,x)+\frac{c_t^2x^2}{2}\frac{\partial^2}{\partial x^2}\phi(t,x)+r_t\phi(t,x)$$
$$+\int_{-1}^{\infty}\left((1+y)\left(\phi\left(t,\frac{x}{1+y}\right)-\phi(t,x)\right)+yx\frac{\partial}{\partial x}\phi(t,x)\right)\bar{\nu}_t(\mathrm{d}y)=0,$$
$$\phi(T,x)=x_+.$$

Then the hedging portfolio in terms of the power-jump assets is given by

$$\phi_t^1 = \frac{B_t}{B_T}\left[\phi\left(t, \frac{U_{t-}}{S_{t-}}\right) - \frac{V_{t-}}{S_{t-}} D_1\phi\left(t, \frac{U_{t-}}{S_{t-}}\right)\right]$$

$$\phi_t^i = \frac{B_t}{B_T}\frac{S_{t-}^i D_1^i\left(S_t\phi\left(t, \frac{U_t}{S_t}\right)\right)}{i!}, \qquad i \geq 2.$$

Fixed a structure preserving martingale measure $Q$, we are going to solve the portfolio optimization problem in the market $M_Q$ by using the "martingale method". Given an initial wealth $w_0 > 0$ and an utility function $U$ we want to find the optimal terminal wealth $\mathcal{W}_T$, that is, the value of $\mathcal{W}_T$ that maximizes $E_P(U(\mathcal{W}_T))$ and can be replicated by a portfolio with initial value $w_0$.

## Definition

A utility function is a mapping $U : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ which is strictly increasing, continuous on $\{U > -\infty\}$, of class $C^\infty$, strictly concave on the interior of $\{U > -\infty\}$ and satisfies

$$U'(\infty) := \lim_{x\to\infty} U'(x) = 0.$$

Denoting by $\mathrm{dom}(U)$ the interior of $\{U > -\infty\}$, we shall consider only two cases:

## Case

*$dom(U) = (0, \infty)$ in which case $U$ satisfies*

$$U'(0) := \lim_{x \to 0^+} U'(x) = \infty.$$

## Case

*$dom(U) = \mathbb{R}$ in which case $U$ satisfies*

$$U'(-\infty) := \lim_{x \to -\infty} U'(x) = \infty.$$

Typical examples for the first case are the so-called HARA utilities $U(x) = \frac{x^{1-p}}{1-p}$ for $p \in \mathbb{R}_+ \backslash \{0, 1\}$, and the logarithmic utility $U(x) = \log(x)$. A typical example for the second case is $U(x) = -\frac{1}{\alpha} e^{-\alpha x}$.

The corresponding Lagrangian to this optimization problem is

$$E_P(U(\mathcal{W}_T)) - \lambda_T E_Q\left(\frac{\mathcal{W}_T}{B_T} - w_0\right) = E_P\left(U(\mathcal{W}_T) - \lambda_T\left(\frac{\mathrm{d}Q_T}{\mathrm{d}P_T}\frac{\mathcal{W}_T}{B_T} - w_0\right)\right)$$

Then, the optimal terminal wealth is given by

$$\mathcal{W}_T = \left(U'\right)^{-1}\left(\frac{\lambda_T}{B_T}\frac{\mathrm{d}Q_T}{\mathrm{d}P_T}\right),$$

where $\lambda_T$ is the solution of the equation

$$E_Q\left[\frac{1}{B_T}\left(U'\right)^{-1}\left(\frac{\lambda_T}{B_T}\frac{\mathrm{d}Q_T}{\mathrm{d}P_T}\right)\right] = w_0.$$

Consider $U(x) = \frac{x^{1-p}}{1-p}$ with $p \in \mathbb{R}_+ \setminus \{0,1\}$. Then we have

$$\mathcal{W}_T = w_0 B_T \frac{\left(\frac{\mathrm{d}P_T}{\mathrm{d}Q_T}\right)^{\frac{1}{p}}}{E_Q\left(\left(\frac{\mathrm{d}P_T}{\mathrm{d}Q_T}\right)^{\frac{1}{p}}\right)} = w_0 B_T \frac{(\xi_T)^{-\frac{1}{p}}}{E_Q\left((\xi_T)^{-\frac{1}{p}}\right)}.$$

where, under some mild conditions on $H(x,t)$,

$$\xi_t = \exp\left(\int_0^t G_s c_s \mathrm{d}\bar{W}_s + \frac{1}{2}\int_0^t G_s^2 c_s^2 \mathrm{d}s + \int_0^t \int_{-\infty}^{+\infty} \log H(s,x) \bar{M}(\mathrm{d}s, \mathrm{d}x) \right.$$
$$\left. - \int_0^t \int_{-\infty}^{+\infty} (H(s,x) - 1 - H(s,x)\log H(s,x))\nu_s(\mathrm{d}x)\mathrm{d}s\right).$$

It is easy to see that the value of the optimal portfolio at time $t$ is just the optimal wealth at time $t$, then

$$\begin{aligned} \mathrm{d}\tilde{\mathcal{W}}_t &= \tilde{\mathcal{W}}_{t-}\left(-\frac{G_t c_t}{p}\mathrm{d}\bar{W}_t + \int_{-\infty}^{+\infty}(e^{-\frac{1}{p}\log H(t,x)} - 1)\bar{M}(\mathrm{d}t,\mathrm{d}x)\right) \\ &= \tilde{\mathcal{W}}_{t-}\left(-\frac{G_t}{p}\frac{\mathrm{d}\tilde{S}_t}{\tilde{S}_{t-}} + \int_{-\infty}^{+\infty}(\frac{1}{H(t,x)^{1/p}} - 1 + \frac{G_t}{p}x)\bar{M}(\mathrm{d}t,\mathrm{d}x)\right) \end{aligned}$$

then

$$\frac{\phi_t^1 S_{t-}}{\mathcal{W}_{t-}} = -\frac{G_t}{p};$$

and we have an optimal portfolio only based in bonds and stocks if and only if

$$H(t,y) = \frac{1}{(1-(G_t/p)y)^p}, \text{with}$$

$$G_t c_t^2 + a_t - r_t + \int_{-\infty}^{\infty} x\left(\frac{1}{(1-(G_t/p)x)^p} - 1\right)\nu_t(\mathrm{d}x) = 0.$$

If another, structure preserving martingale, is chosen by the market, then the optimal portfolio will contain derivatives that, in terms of the power assets, will be given by

$$\phi_t^i = \frac{\mathcal{W}_{t-}}{i! B_t} \frac{\partial_i}{\partial y_i} \frac{1}{H(t,y)}_{|y=0}, \quad i \geq 2$$

where we assume also that, fixed $t$, $H(t,y)$ is an analytic function and that

$$\sum_{i=2}^{\infty} \frac{|m_t|_i}{i!} \frac{\partial^i}{\partial y^i} \frac{1}{H(t,y)}_{|y=0} < \infty,$$

for all $0 \leq t \leq T$, where

$$|m_t|_i = \int_{-\infty}^{+\infty} |y|^i \bar{\nu}_t(\mathrm{d}y)$$

In order to replicate $\mathcal{W}_T$ we need to know its price process function and this depends on the utility considered:

$$E_Q\left[\frac{B_t}{B_T}\left(U'\right)^{-1}\left(\frac{\lambda_T}{B_T}\xi_T\right)|\mathcal{F}_t\right]$$

Suppose now that the utility function verifies

$$(U')^{-1}(xy) = a_1(x)(U')^{-1}(y) + a_2(x), \text{ for any } x, y \in (0, \infty),$$

for certain $C^\infty$ functions $a_1(x), a_2(x)$. Then, it is easy to see that the price function of $\mathcal{W}_T$ verifies

$$E_Q\left[\frac{B_t}{B_T}\mathcal{W}_T|\mathcal{F}_t\right] = \varphi(t, T)\mathcal{W}_t + \chi(t, T)$$

## Lemma

*$(U')^{-1}(xy) = a_1(x)(U')^{-1}(y) + a_2(x)$, for any $x, y \in (0, \infty)$ if and only if $\frac{U'(x)}{U''(x)} = ax + b$, for any $x \in \mathrm{dom}(U)$, for some $a, b \in \mathbb{R}$.*

These utility functions include HARA and exponential utilities as particular cases. For these class of utility functions we can obtain similar results

In fact if $U'(x)/U''(x) = ax + b$ and

$$E_Q(\frac{B_t}{B_T}|\mathcal{F}_T) = \varphi(t,T)\mathcal{W}_t + \chi(t,T),$$

then

$$\phi_t^1 = G_t \frac{\varphi(t,T)(a\mathcal{W}_{t-} + b)}{S_{t-}}$$

$$\phi_t^i = \frac{\varphi(t,T)}{i!B_t}\frac{\partial^i}{\partial y^i}\Big((U')^{-1}(U'(\mathcal{W}_{t-})H(t,y))\Big)_{|y=0}, \quad i \geq 2$$

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