Joint conditional density of a Markov process and its local time with applications to default risk modelling

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Preliminaries

- Suppose the default time of a certain firm is modelled by some τ which is a positive random variable defined on $(\Omega, \mathcal{H}, \mathbb{P})$.
- Standard assumption is that $\mathbb{P}(\tau = 0) = 0$ and $\mathbb{P}(\tau > t) > 0$ for all $t \in \mathbb{R}_+$.
- There is a reference filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ that models the information obtained from relevant (or irrelevant) asset prices, news, accounting information, etc. Typically τ is not an $\mathcal{F}$-stopping time.
- Since default is a public event, however, we define a new filtration $\mathcal{G}$ which is obtained by enlarging $\mathcal{F}$ just enough so that τ becomes a $\mathcal{G}$-stopping time.

Hazard process of a random time

- We are interested in $\mathbb{P}(\tau > T|\mathcal{G}_t)$, which would give a price for the defaultable zero-coupon bond provided $\mathbb{P}$ is some risk-neutral measure.
- The key formula (due to Dellacherie) is that for any $Y \in \mathcal{H}$

$$\mathbb{E}[\mathbf{1}_{[\tau>t]}Y|\mathcal{G}_s] = \mathbf{1}_{[\tau>s]}\frac{\mathbb{E}[\mathbf{1}_{[\tau>t]}Y|\mathcal{F}_s]}{\mathbb{P}(\tau > s|\mathcal{F}_s)},$$

  for $s \leq t$.
- Define the supermartingale Z by $Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$. ($Z$ is said to be the *Azema's supermartingale* associated to τ.)
- Let $\Gamma_t = -\log Z_t$. Γ is the *hazard process* associated to τ.

Martingales associated with Γ

- There exists a unique $\mathcal{F}$-predictable and increasing process, A, with $A_0 = 0$ such that $Z + A$ is an $\mathcal{F}$-martingale.
- Define M by $M_t = \mathbf{1}_{[\tau > t]} e^{\Gamma_t}$. Then, M is a $\mathcal{G}$-martingale. Moreover, for any $\mathcal{F}$-martingale m, mM is a $\mathcal{G}$-martingale.
- Let $N_t = \mathbf{1}_{[\tau > t]}$. Then $N + \Lambda$ is a $\mathcal{G}$-martingale where Λ is defined by $d\Lambda = \mathbf{1}_{[t \leq \tau]} \frac{dA_t}{Z_{t-}}$.

Valuation of defaultable bonds

- Most interesting case is when the compensator of N is absolutely continuous, i.e. $d\Lambda_t = \mathbf{1}_{[t \leq \tau]} \lambda_t dt$. Suppose this is the case and let $S_t := \mathbb{P}(\tau > T | \mathcal{G}_t)$.
- Let

$$V_t := \mathbb{E}\left[\exp\left(-\int_t^T \lambda_u\right) du \middle| \mathcal{G}_t\right]. \tag{1}$$

Duffie, Schroder and Skiadas (1996) have proved that

$$S_t = V_t - \mathbb{E}[\Delta V_\tau | \mathcal{G}_t],$$

on the set $[\tau > t]$.

A simplifying assumption

- Often the following martingale invariance property is assumed in credit risk models:

  (**H**) Every square integrable $\mathcal{F}$-martingale is a $\mathcal{G}$-martingale.

- The **H**-Hypothesis above is equivalent to

$$\mathbb{P}(\tau \leq s|\mathcal{F}_t) = \mathbb{P}(\tau \leq s|\mathcal{F}_\infty),$$

  for every $s \leq t$.
- This implies Z has a modification that is decreasing.
- If we further assume that Z is predictable, this implies $A = 1 - Z$.
- If Z is absolutely continuous, then so is Λ and the process V in (1) does not jump at τ.

- Suppose $\mathcal{F}$ is the filtration generated by some traded risky asset, whose price process is denoted with S and which is subject to default. Assume further that the market is arbitrage free and *complete* given the filtration $\mathcal{F}$. Blanchet-Scalliet and Jeanblanc (2004) show that if the market remains arbitrage free when the filtration is enlarged to $\mathcal{G}$, then there exists some equivalent martingale measure for S under which **H**-Hypothesis holds.
- The completeness assumption is crucial for the above result. In Duffie and Lando (2000) the market is arbitrage-free yet the **H**-Hypothesis does not hold.
- Kusuoka (1999) also presents an example of an arbitrage free market where the **H**-Hypothesis is not satisfied. In particular he shows **H**-Hypothesis is not stable under a change of measure.

Brownian local time

Let W be a standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Define for $a < 0$,

$$\tau_a := \inf\{t > 0 : W_t = a\}.$$

Let L^x be the local time process of W at level $x \in \mathbb{R}$ which could be defined by the following a.s. limit:

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[x \leq W_s \leq x+\varepsilon]} ds. \tag{2}$$

Local time properties

L^x has the following properties:

- It satisfies for each $t \geq 0$

$$|W_t - x| - |x| = \int_{0+}^{t} \operatorname{sgn}(W_s - x) dW_s + L_t^x. \tag{3}$$

- L^x is continuous and increasing for each x.
- **(Occupation times formula)** For any bounded Borel measurable g

$$\int_{-\infty}^{\infty} L_t^x g(x) dx = \int_0^t g(W_s) ds.$$

- $L_t^0 > 0$ for all $t > 0$.

Probability distribution for local times

It is well known that

$$\mathbb{P}(W_t \in dw, L_t^x \in dy) = \frac{1}{\sqrt{2\pi t^3}}\psi(t,w,y)dwdx, \tag{4}$$

where

$$\psi(t,w,y) := (|a| + |w-a| + y)\exp\left(-\frac{(|a|+|w-a|+y)^2}{2t}\right),$$

for $y > 0$ and $w \in \mathbb{R}$. By integrating above we get

$$\mathbb{P}(L_t^x \le y) = 2\Phi\left(\frac{y+|x|}{\sqrt{t}}\right) - 1,$$

where Φ is the cumulative probability distribution function of standard normal.

In particular, recall $a < 0$,

$$
\begin{aligned}
\mathbb{P}(L_t^a = 0) &= 2\Phi\left(-\frac{a}{\sqrt{t}}\right) - 1 \\
&= 1 - 2\Phi\left(\frac{a}{\sqrt{t}}\right) \\
&= 1 - 2\mathbb{P}(W_t \leq a) = \mathbb{P}(\tau_a > t),
\end{aligned}
$$

due to the reflection principle of brownian motion.

This is not a coincidence!

- It follows from (3) that $[\tau_a > t] \subset [L_t^a = 0]$
- Therefore, $[L_t^a = 0] = [L_t^a = 0, \tau_a < t] \cup [\tau_a > t]$.
- Next, use the strong Markov property of W and that $L_t^0 > 0$ for all $t > 0$ to conclude $[L_t^a = 0] = [\tau_a > t]$.

Observing W through an auxiliary process

Let B be another brownian motion independent of W. Let Y be the strong solution to the following SDE

$$dY_t = \alpha(t, W_t, Y_t)dt + dB_t, \qquad Y_0 = 0, \tag{5}$$

for each $t \in [0, T]$ where α is Lipschitz and $T > 0$ is a constant. We further make the following assumption on α.

Assumption 1

$\mathbb{E}\left[\int_0^T \alpha^2(t, W_t, Y_t)dt\right] < \infty.$

Survival probabilities

- We are interested in the *survival probability* $Z_t := \mathbb{P}[\tau > t | \mathcal{F}_t^Y]$, for each $t \in [0, T]$ where $\mathcal{F}^Y$ is the minimal filtration generated by Y satisfying usual hypotheses. Z is a supermartingale with a càdlàg modification, which we'll use henceforth.
- Next let $\zeta_t := \mathbb{P}[L_t^a = 0 | \mathcal{F}_t^Y], t \geq 0$, and observe that $\zeta_t = Z_t$, a.s. for each t.
- It can be checked that ζ also admits a càdlàg modification. Thus, we may conclude ζ and Z are indistinguishable.

Joint conditional law of W and L^a

In order to find the joint conditional law of W and L^a, we'll first find that of W and X^ε where

$$X_t^\varepsilon := \frac{1}{\varepsilon}\int_0^t \mathbf{1}_{[a \le W_s \le a+\varepsilon]} ds, \tag{6}$$

converges to L_t^a, a.s.. Let $\mathbb{R}_{++}$ stand for the strictly positive real numbers and $f : \mathbb{R}_{++} \times \mathbb{R}$ be twice continuously differentiable with respect to both parameters. Define $f_t := f(X_t^\varepsilon, W_t), f_t^{(x)} := \frac{\partial}{\partial x} f(X_t^\varepsilon, W_t), f_t^{(ww)} := \frac{\partial^2}{\partial w^2} f(X_t^\varepsilon, W_t)$ and let $\pi_t(f) := \mathbb{E}[f_t | \mathcal{F}_t^Y]$.

Standard results on optimal filtering yield the following

Lemma 1

Let Y satisfy (5) and define $\alpha_t := \alpha(t, W_t, Y_t)$. Then

$$\begin{aligned}\pi_t(f) &= f_0 + \int_{0+}^{t} \left\{ \frac{1}{\varepsilon} \pi_s \left(f^{(x)} \mathbf{1}_{[a \leq W \leq a+\varepsilon]} \right) + \frac{1}{2} \pi_s \left(f^{(ww)} \right) \right\} ds \\ &\quad + \int_{0+}^{t} \pi_s(f\alpha) - \pi_s(f)\pi_s(\alpha) dB_s^Y,\end{aligned}$$

where B^Y is an $\mathcal{F}^Y$-Brownian motion defined by

$$dB_t^Y = dY_t - \pi_t(\alpha)dt.$$

SPDEs satisifed by the joint laws

- Let
$$g_t^{\varepsilon W}(x,w) := \mathbb{P}(X_t^\varepsilon \in dx, W_t \in dw | \mathcal{F}_t^Y)/dxdw,$$
for $x \in \mathbb{R}_{++}$ and $w \in \mathbb{R}$.
- Looking at (4) reveals that in general we don't expect $g^{\varepsilon W}$ to be differentiable with respect to its second parameter. Therefore, the partial derivatives in the next theorem should be understood in the sense of generalized functions.
- Let $\mathcal{D}$ be the set of infinitely differentiable functions over $\mathbb{R}_{++} \times \mathbb{R}$ with a bounded support. One can view $g_t^{\varepsilon W}$ as a generalized function on $\mathcal{D}$ depending on a random parameter.

Lemma 2

Let $\bar{\alpha}_t(w, Y_t) := \alpha(t, w, Y_t) - \pi_t(\alpha)$. *Assume*

$$\mathbb{E}\left[\int_0^T \int_{-\infty}^{\infty} \int_0^{\infty} \left(g_t^{\varepsilon W}(x, w)\bar{\alpha}_t(w, Y_t)\right)^2 dxdwdt\right] < \infty. \tag{7}$$

Let

$$I_t = \int_0^t \int_0^{\infty} \int_{-\infty}^{\infty} f(x, w) g_s^{\varepsilon W}(x, w)\bar{\alpha}_s(w, Y_s) dwdxdB_s^Y.$$

Then, for each $t \in [0, T]$ *and for all* $f \in \mathcal{D}$

$$\begin{aligned} I_t &= \int_0^{\infty} \int_{-\infty}^{\infty} \int_0^t f(x, w) g_s^{\varepsilon W}(x, w)\bar{\alpha}_s(w, Y_s) dB_s^Y dwdx \\ &= \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^t f(x, w) g_s^{\varepsilon W}(x, w)\bar{\alpha}_s(w, Y_s) dB_s^Y dxdw. \end{aligned}$$

Theorem 3

Under the assumptions of Lemma 2, for $(x, w) \in \mathbb{R}_{++} \times \mathbb{R}$,

$$\begin{aligned} g_t^{\varepsilon W}(x,w) &= g_0^{\varepsilon W}(x,w) \\ &- \int_0^t \left\{ \frac{1}{\varepsilon} \frac{\partial}{\partial x} g_s^{\varepsilon W}(x,w) \mathbf{1}_{[a \le w \le a+\varepsilon]} + \frac{1}{2} \frac{\partial^2}{\partial w^2} g_s^{\varepsilon W}(x,w) \right\} ds \\ &+ \int_{0+}^t g_s^{\varepsilon W}(x,w) \left(\alpha(s,w,Y_s) - \pi_s(\alpha)\right) dB_s^Y, \end{aligned}$$

where the derivatives are understood in the sense of generalized functions.

As a corollary to the above theorem we get the conditional law of L^a *and* (L^a, W).

SPDE satisfied by the conditional law of (L^a, W)

Corollary 4

Let

$$\begin{aligned} g_t^L(x) &:= \mathbb{P}[L_t^a \in dx | \mathcal{F}_t^Y]/dx, \ \text{for } x \in \mathbb{R}_{++}, \ \text{and} \\ g_t^{LW}(x,w) &:= \mathbb{P}[L_t^a \in dx, W_t \in dw | \mathcal{F}_t^Y]/dxdw, \ \text{for } (x,w) \in \mathbb{R}_{++} \times \mathbb{R}. \end{aligned}$$

Then,

$$\begin{aligned} g_t^L(x) &= g_0^L(x) - \int_{0+}^t \frac{\partial}{\partial x} g_s^{LW}(x,a) ds \\ &+ \int_{0+}^t \int_{-\infty}^{\infty} g_s^{LW}(x,w) \left(\alpha(s,w,Y_s) - \pi_s(\alpha)\right) dwdB_s^Y; \ \text{and} \end{aligned} \tag{8}$$

$$
\begin{aligned}
g_t^{LW}(x,w) &= g_0^{LW}(x,w) \\
&\quad - \int_{0+}^{t} \frac{\partial}{\partial x} g_s^{LW}(x,a)\delta_a(w)ds + \frac{1}{2}\int_{0+}^{t} \frac{\partial^2}{\partial w^2} g_s^{LW}(x,w)ds \\
&\quad + \int_{0+}^{t} g_s^{LW}(x,w)\left(\alpha(s,w,Y_s) - \pi_s(\alpha)\right) dB_s^Y, \qquad (9)
\end{aligned}
$$

where $\frac{\partial}{\partial x} g_s^{LW}(x,a)\delta_a(w)$ is the direct product of the generalized function $\frac{\partial}{\partial x} g_s^{LW}(\cdot,a)$ with the delta function at a.

- Note that when $\alpha(t,\cdot,y)$ is constant for all t and y, $\alpha_t = \pi_t(\alpha)$ for each t. In this case (9) reduces to

$$\begin{aligned} g_t^{LW}(x,w) &= g_0^{LW}(x,w) - \int_{0+}^t \frac{\partial}{\partial x} g_s^{LW}(x,a)\delta_a(w)ds \\ &+ \frac{1}{2}\int_{0+}^t \frac{\partial^2}{\partial w^2} g_s^{LW}(x,w)ds. \end{aligned}$$

- It can be shown by direct manipulation that (4) satisfies above. This is no surprise since with this particular choice of α, Y becomes independent of W.

General case

Now let θ be a strong solution of

$$d\theta_t = \mu(t, \theta_t)dt + \sigma(t, \theta_t)dW_t, \tag{10}$$

with $\theta_0 = 0$, where μ and σ are deterministic Lipschitz functions. The observation process, Y, is, again, given by

$$dY_t = \alpha(t, \theta_t, Y_t)dt + dB_t. \tag{11}$$

We now assume that the quadratic covariation of B and W satisfies

$$\frac{d}{dt}[B, W]_t = \rho(t, \theta_t, Y_t), \tag{12}$$

for a deterministic function ρ. In particular, $\rho \equiv 0$ if B and W are independent. We suppose $\tau := \inf\{t > 0 : \theta_t = a\}$, for $a < 0$ is finite a.s. so that we are not dealing with a vacuous problem.

- Analogously, let

$$X_t^\varepsilon = \frac{1}{\varepsilon}\int_{0+}^t \mathbf{1}_{[a\le\theta_s\le a+\varepsilon]}\sigma^2(s,\theta_s)ds.$$

  Similarly, X_t^ε converges to L_t^a a.s. for every t, where L^a is the local time of θ at a.
- We next introduce some notation:

Let

$$\begin{aligned}
g_t^L(x) &:= \mathbb{P}[L_t^a \in dx|\mathcal{F}_t^Y]/dx, \text{ for } x \in \mathbb{R}_{++}, \\
g_t^{L\theta}(x,\theta) &:= \mathbb{P}[L_t^a \in dx, \theta_t \in d\theta|\mathcal{F}_t^Y]/dxd\theta, \text{ for } (x,\theta) \in \mathbb{R}_{++}\times\mathbb{R}, \text{ and} \\
g_t^{\varepsilon\theta}(x,\theta) &:= \mathbb{P}[X_t^\varepsilon \in dx, \theta_t \in d\theta|\mathcal{F}_t^Y]/dxd\theta, \text{ for } (x,\theta) \in \mathbb{R}_{++}\times\mathbb{R}.
\end{aligned}$$

We *assume* that *all* the above densities exist. Assuming further $\sigma(t,\cdot) \in C^2, \rho(t,\cdot,y) \in C^1$, and $\mu(t,\cdot) \in C^1$, set

$$
\begin{aligned}
\mathcal{L}^* g_t^{\varepsilon\theta}(x,\theta) &= -\frac{1}{\varepsilon}\mathbf{1}_{[a\le\theta\le a+\varepsilon]}\sigma^2(t,\theta)\frac{\partial}{\partial x}g_t^{\varepsilon\theta}(x,\theta) - \frac{\partial}{\partial\theta}\left[g_t^{\varepsilon\theta}(x,\theta)\mu(t,\theta)\right] \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\left[g_t^{\varepsilon\theta}(x,\theta)\sigma^2(t,\theta)\right] \\
\mathcal{L}^* g_t^{L\theta}(x,\theta) &= -\delta_a(\theta)\sigma^2(t,a)\frac{\partial}{\partial x}g_t^{L\theta}(x,a) - \frac{\partial}{\partial\theta}\left[g_t^{L\theta}(x,\theta)\mu(t,\theta)\right] \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial\theta^2}\left[g_t^{L\theta}(x,\theta)\sigma^2(t,\theta)\right] \\
\mathcal{N}^* g_t^{\varepsilon\theta}(x,\theta) &= -\frac{\partial}{\partial\theta}\left[g_t^{\varepsilon\theta}(x,\theta)\rho(t,\theta,Y_t)\sigma(t,\theta)\right] \\
\mathcal{N}^* g_t^{L\theta}(x,\theta) &= -\frac{\partial}{\partial\theta}\left[g_t^{L\theta}(x,\theta)\rho(t,\theta,Y_t)\sigma(t,\theta)\right]
\end{aligned}
$$

Analogous results in the general case

Theorem 5

Let θ and Y be as in (10) and (11), respectively. Suppose $\sigma(t,\cdot) \in C^2, \rho(t,\cdot,y) \in C^1$, and $\mu(t,\cdot) \in C^1$ and that, for each $\varepsilon > 0$,

$$\int_0^t \int \int \left| f(x,\theta)\mathcal{L}^* g_s^{\varepsilon\theta}(x,\theta) \right| dx d\theta ds < \infty, \text{ and}$$

$$\mathbb{E}[\int_0^t \int \int f^2(x,\theta) \quad \{ \quad \mathcal{N}^* g_s^{\varepsilon\theta}(x,\theta) \\ + g_s^{\varepsilon\theta}(x,\theta)(\alpha(s,\theta,Y_s) - \pi_s(\alpha))\}^2 dx d\theta ds] < \infty,$$

for all $f \in \mathcal{D}$ and $t > 0$.

Then,

$$
\begin{aligned}
g_t^{\varepsilon\theta}(x,\theta) &= g_0^{\varepsilon\theta}(x,\theta) + \int_0^t \mathcal{L}^* g_s^{\varepsilon\theta}(x,\theta)ds \\
&\quad + \int_0^t \left\{\mathcal{N}^* g_s^{\varepsilon\theta}(x,\theta) + g_s^{\varepsilon\theta}(x,\theta)\left(\alpha(s,\theta,Y_s) - \pi_s(\alpha)\right)\right\} dB_s^Y,
\end{aligned}
$$

where $dB_s^Y = dY_s - \pi_s(\alpha)ds$.

Corollary 6

Under the assumptions of Theorem 5

$$
\begin{aligned}
g_t^{L\theta}(x,\theta) &= g_0^{L\theta}(x,\theta) + \int_0^t \mathcal{L}^* g_s^{L\theta}(x,\theta)ds \\
&\quad + \int_0^t \left\{ \mathcal{N}^* g_s^{L\theta}(x,\theta) + g_s^{L\theta}(x,\theta)\left(\alpha(s,\theta,Y_s) - \pi_s(\alpha)\right)\right\} dB_s^Y, \\
g_t^L(x) &= g_0^L(x) - \int_{0+}^t \sigma^2(s,a)\frac{\partial}{\partial x} g_s^{L\theta}(x,a)ds \\
&\quad + \int_{0+}^t \int_{-\infty}^{\infty} g_s^{L\theta}(x,\theta)\left(\alpha(s,\theta,Y_s) - \pi_s(\alpha)\right) d\theta dB_s^Y.
\end{aligned}
$$

A special case

Our assumption is that the relevant conditional densities exist. The following theorem shows a situation when this is the case.

Theorem 7

Let θ satisfy

$$d\theta_t = dW_t + \mu(t, \theta_t)dt,$$

such that B and W are independent. Suppose $\int_0^T \mu^2(t, \theta_t)dt < \infty$. Then $g^L, g^{\varepsilon\theta}$, and $g^{L\theta}$ exist.

Some examples

Although the above case in which the densities exist seems very special, in many practical applications, as long as we are only interested in finding the default probabilities, the problem can be reduced to this case after simple transformations.

Example 8

Let V be a geometric Brownian motion, i.e. $V_t = V_0 \exp\left(\sigma W_t + \frac{1}{2}(\mu - \sigma^2)t\right)$, with $V_0 > 0$ and if τ is the first time that θ falls below some level c such that $V_0 > c > 0$ then $\tau = \inf\{t > 0 : \sigma W_t + \frac{1}{2}(\mu - \sigma^2)t = a\}$, where $a = \log \frac{c}{V_0}$.

Example 9

Suppose V satisfies

$$dV_t = \mu(t, V_t)dt + \sigma(t, V_t)dW_t,$$

with $V_0 = 0$ and suppose further that $\sigma > \varepsilon > 0$ for some ε, and that default occurs at τ which is the first time that V hits $a < 0$. Let

$$F(t,x) = a + \int_a^x \sigma^{-1}(t,v)dv,$$

and define $\theta_t = F(t, V_t)$. Note that $\tau = \inf\{t > 0 : \theta_t = a\}$. Moreover,

$$d\theta_t = dW_t + \left\{F_t(t,\theta_t) + \frac{\mu(t,\theta_t)}{\sigma(t,\theta_t)} - \frac{1}{2}\frac{\sigma_x(t,\theta_t)}{\sigma^2(t,\theta_t)}\right\}dt,$$

where $F_t := \frac{\partial F}{\partial t}(t,x)$ and $\sigma_x := \frac{\partial \sigma}{\partial x}(t,x)$. Under appropriate integrability conditions on μ, σ, σ_x and F_t, one would be able to satisfy the conditions of Theorem 7.

Applications to default risk

- Consider a company which issues a bond with a face value of \$1 and maturity $T > 0$.
- Let θ be a proxy for the firm value such that

$$d\theta_t = \sigma(\theta_t)dW_t + \mu(\theta_t)dt, \tag{13}$$

  with $\theta_0 = 0$ and $\sigma : \mathbb{R} \mapsto \mathbb{R}$ and $\mu : \mathbb{R} \mapsto \mathbb{R}$ are Lipschitz.
- Suppose $\sigma(\cdot) > 0$.
- Let τ be the first time that θ falls below $a < 0$. The firm will default and won't make payments if $\tau \leq T$.

Assumption 2

There exists a $\mathbb{Q} \sim \mathbb{P}$ under which θ is a square integrable martingale.

- θ is not publicly observable. However, the market observes Y which satisfies
$$dY_t = dB_t + \alpha(t, \theta_t, Y_t),$$
where B is a brownian motion independent of W.
- Under the assumption above $[L_t^a = 0] = [\tau > t]$, a.s.; thus, $Z_t = \mathbb{P}[\tau > t | \mathcal{F}_t^Y] = \mathbb{P}[L_t^a = 0 | \mathcal{F}_t^Y]$.

$\mathcal{F}^Y$-Doob-Meyer decomposition for Z

- Under the assumptions of Theorem 5, it follows from Corollary 6 that

$$
\begin{aligned}
Z_t &= 1-\sigma^2(a)\int_{0+}^t g_s^{L\theta}(0+,a)ds \\
&\quad - \int_{0+}^t \mathbb{E}[\mathbf{1}_{[L_s>0]}(\alpha_s-\pi_s(\alpha))|\mathcal{F}_s^Y]dB_s^Y \\
&= 1-\sigma^2(a)\int_{0+}^t g_s^{L\theta}(0+,a)ds \\
&\quad - \int_{0+}^t \mathbb{E}[\mathbf{1}_{[\tau>s]}(\alpha_s-\pi_s(\alpha))|\mathcal{F}_s^Y]dB_s^Y,
\end{aligned}
$$

  where $g_s^{L\theta}(0+,a) := \lim_{x\downarrow 0} g_s^{L\theta}(x,a)$.
- The above shows $Z+A$, where $A_t=\sigma^2(a)\int_{0+}^t g_s^{L\theta}(0+,a)ds$, is an $\mathcal{F}^Y$-martingale.

Price of defaultable bonds

- Of course, the market also observes whether the default event has occurred or not. In other words $\mathcal{G}$ with $\mathcal{G}_t = \mathcal{F}_t^Y \vee \{\tau \wedge t\}$ defines the market's filtration. Assuming zero interest rates, the fair price S of this defaultable bond is given by $S_t = \mathbf{1}_{[\tau>t]}\mathbb{E}[\mathbf{1}_{[\tau>T]}|\mathcal{G}_t]$. Let $Z_t = \mathbb{P}[\tau > t|\mathcal{F}_t^Y]$ for each t. Then, it is well known that

$$\mathbf{1}_{[\tau>t]}\mathbb{E}[\mathbf{1}_{[\tau>T]}|\mathcal{G}_t] = \mathbf{1}_{[\tau>t]}Z_t^{-1}\mathbb{E}[Z_T|\mathcal{F}_t^Y]. \tag{14}$$

- Under the **(H)**-Hypothesis

$$S_t = \mathbf{1}_{[\tau>t]}\mathbb{E}\left[\exp\left(\sigma^2(a)\int_t^T \frac{g_u^{L\theta}(0+,a)}{Z_u}du\right)|\mathcal{G}_t\right].$$

An example where **(H)**-Hypothesis is satisfied

Suppose Y is of the form

$$dY_t = dB_t + Y_t dt,$$

i.e. $\alpha(t, \theta, y) = y$. Then, obviously, $\alpha_s = \pi_s(\alpha) = Y_s$. Thus, **(H)**-Hypothesis is satisfied. This can be seen either by looking at the decomposition for Z above or by noticing Y is independent of W, hence, τ.

An example where **(H)**-Hypothesis is not satisfied

Suppose $\alpha : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ is defined by for $a < 0$

$$\alpha(t, \theta, y) = \begin{cases} 0, & \text{if } \theta \geq a; \\ (\theta - a)^2, & \text{if } \theta < a. \end{cases}$$

Let $\theta = W$ and τ as above. Suppose Y satisfies

$$dY_t = dB_t + \alpha(t, \theta_t, Y_t)dt.$$

If **(H)**-Hypothesis were satisfied, then B^Y would be a $\mathcal{G}$-martingale.

Consequently, β defined by $\beta_t = B^Y_{t\wedge\tau}$ would be a $\mathcal{G}$-Brownian motion. Theory of progressive enlargement of filtrations yields

$$\beta_t = R_t + \int_0^t \mathbf{1}_{[\tau \geq s]} \frac{\mathbb{E}[\mathbf{1}_{[\tau > s]}(\alpha_s - \pi_s(\alpha))|\mathcal{F}^Y_s]}{Z_s} ds,$$

where R is a $\mathcal{G}$-martingale. Due to the particular choice for α

$$\mathbb{E}[\mathbf{1}_{[\tau > s]}(\alpha_s - \pi_s(\alpha))|\mathcal{F}^Y_s] = -Z_s \mathbb{E}[\mathbf{1}_{\{W_t < a\}}(W_t - a)^2|\mathcal{F}^Y_t].$$

Therefore,

$$\beta_t = R_t - \int_0^t \mathbf{1}_{[\tau \geq s]} \mathbb{E}[\mathbf{1}_{\{W_s < a\}}(W_s - a)^2|\mathcal{F}^Y_s] ds.$$

This shows β is a not a $\mathcal{G}$-martingale since $\mathbb{E}[\mathbf{1}_{\{W_t < a\}}(W_t - a)^2|\mathcal{F}^Y_t]$ is strictly positive a.s. for each $t > 0$.

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