Dam Rain and Cumulative Gain

Dorje C. Brody

Reader in Mathematics
Department of Mathematics,
Imperial College, London SW7 2BZ
http://www.imperial.ac.uk/people/d.brody

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Work based on:

- D. C. Brody, L. P. Hughston, & A. Macrina (2007) "Dam Rain and Cumulative Gain" (Submitted)

Other related papers:

- D. C. Brody, L. P. Hughston, & A. Macrina (2007) "Beyond hazard rates: a new framework for credit-risk modelling" In *Advances in Mathematical Finance, Festschrift volume in honour of Dilip Madan* R. Elliott, M. Fu, R. Jarrow, and Ju-Yi Yen, eds. (Basel: Birkhäuser).

- D. C. Brody, L. P. Hughston, & A. Macrina (2007) "Information-based approach to asset pricing" (Submitted)

- M. Yor (2007) Some remarkable properties of gamma processes. In *Advances in Mathematical Finance, Festschrift volume in honour of Dilip Madan* R. Elliott, M. Fu, R. Jarrow, and Ju-Yi Yen, eds. (Basel: Birkhäuser).

Cumulative gains with gamma information

There are problems in finance and insurance that involve the analysis of accumulation processes—processes representing cumulative gains or losses.

The typical setup is as follows: we fix an accounting period $[0, T]$.

At time T a contract pays a random cash flow X_T given by the terminal value of a process of accumulation.

In the case of an insurance contract, the random X_T represents the totality of the payments made at T in settlement of claims arising over the period $[0, T]$.

The problem facing the insurance firm is the valuation of the random cash flow.

We write $\{S_t\}$ for the value process of the contract that pays X_T at T, $\{\mathcal{F}_t\}$ for the market filtration, and $\mathbb{Q}$ for the pricing measure established by market.

Then the value at t of the contract that pays X_T at T is

$$S_t = P_{tT}\mathbb{E}[X_T|\mathcal{F}_t], \tag{1}$$

where $\mathbb{E}[-] = \mathbb{E}^{\mathbb{Q}}[-]$ and P_{tT} is the discount factor (assumed deterministic).

One can interpret S_t as the reserve that the insurance firm requires at t in order to ensure that X_T will be payable at T.

Similarly, the cost I_{tT} at t of a simple stop-loss reinsurance contract that pays out $(X_T - K)^+$ at T for some fixed threshold K is given by

$$I_{tT} = P_{tT}\mathbb{E}[(X_T - K)^+|\mathcal{F}_t]. \tag{2}$$

We shall assume that $\{\mathcal{F}_t\}$ is generated by an aggregate claims process $\{\xi_t\}$, where for each t the random variable ξ_t represents the totality of claims known at t to be payable at T.

Problem: given the history of claims up to time t, what is the appropriate reserve to be set aside for settlement of these and any future claims?

To obtain a specific solution to the problem we need to specify the aggregate claims process $\{\xi_t\}$ and the measure $\mathbb{Q}$, then work out the reserve process $\{S_t\}$.

Once we have $\{S_t\}$, we can value various types of reinsurance contracts.

The purpose of this talk is to present a modelling framework for accumulation processes, and to establish explicit pricing formulae for various contracts.

In particular, we shall assume that $\{\xi_t\}$ takes the form

$$\xi_t = X_T \gamma_{tT}, \tag{3}$$

where $\{\gamma_{tT}\}$ is a gamma bridge over the interval $[0, T]$, independent of X_T.

The motivation for the specific form of the accumulation process arises partly from the idea that the *gamma process* can be used as a mathematical basis for describing the aggregate losses associated with insurance claims.

This idea dates back to the work of Hammersley (1955), Moran (1956), Gani (1957), and Kendall (1957) in connection with the theory of storage and dams.

Moran, in particular, presented an argument showing that the aggregate amount of rainfall accumulating in a dam can be modelled by a gamma process.

Gani pointed out the relevance to insurance, the argument being that providing that the portfolio of events insured is sufficiently large, one can think of the arrival of claims as being analogous to the accumulation of dam rain.

Before we proceed, however, let us begin with a brief introduction to gamma processes and associated bridge processes.

Gamma processes and associated martingales

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.

By a standard gamma process $\{\gamma_t\}_{0 \leq t \leq \infty}$ with rate m we mean a process with independent increments such that $\gamma_0 = 0$ and such that the random variable $\gamma_u - \gamma_t$ for $u \geq t \geq 0$ has a gamma distribution with parameter $m(u-t)$:

$$g(x) = \mathbb{1}_{\{x>0\}} \frac{x^{m(u-t)-1} \mathrm{e}^{-x}}{\Gamma[m(u-t)]} \tag{4}$$

It follows from $\Gamma[a+1]/\Gamma[a] = a$ that

$$\mathbb{E}[\gamma_t] = mt, \tag{5}$$

which justifies the interpretation of the parameter m as the *mean growth rate* of the process.

A straightforward calculation shows that

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda\gamma_t}\right] = \frac{1}{(1-\mathrm{i}\lambda)^{mt}} \tag{6}$$

for $t \geq 0$ and all $\lambda \in \mathbb{C}$ such that $\mathrm{Im}(\lambda) > -1$.

An alternative expression for the characteristic function is given by the Lévy-Khinchine representation $\mathbb{E}\left[e^{i\lambda\gamma_t}\right] = e^{-t\psi(\lambda)}$, where

$$\psi(\lambda) = m\ln(1 - i\lambda) = \int_0^\infty mx^{-1}e^{-x}\left(1 - e^{i\lambda x}\right)dx, \tag{7}$$

which shows that the associated Lévy density is given by $mx^{-1}e^{-x}$.

From the independent increments property we deduce that

$$\{\gamma_t - mt\} \quad \text{and} \quad \{\gamma_t^2 - 2mt\gamma_t + mt(mt - 1)\} \tag{8}$$

are martingales.

Likewise, $\{(1 + \alpha)^{mt}e^{-\alpha\gamma_t}\}$ is a geometric gamma martingale.

In general, we let $\{L_n^{(k)}(z)\}$ denotes the associated Laguerre polynomials:

$$(1 + \alpha)^{mt}e^{-\alpha z} = \sum_{n=0}^\infty L_n^{(mt-n)}(z)\,\alpha^n. \tag{9}$$

Then $\{L_n^{(mt-n)}(\gamma_t)\}_{n=0,1,\ldots,\infty}$ are also martingales.

Properties of the gamma bridge process

Now suppose $\{\gamma_t\}_{0 \le t \le \infty}$ is a standard gamma process with rate m.

For fixed T define the process $\{\gamma_{tT}\}_{0 \le t \le T}$ by

$$\gamma_{tT} = \frac{\gamma_t}{\gamma_T}. \tag{10}$$

Then clearly we have $\gamma_{0T} = 0$ and $\gamma_{TT} = 1$.

We refer to $\{\gamma_{tT}\}$ as the standard gamma bridge over the interval $[0, T]$ associated with the gamma process $\{\gamma_t\}$.

The density function of the random variable γ_{tT} is given by

$$f(y) = \mathbb{1}_{\{0 < y < 1\}} \frac{y^{mt-1}(1-y)^{m(T-t)-1}}{\mathrm{B}[mt, m(T-t)]}, \tag{11}$$

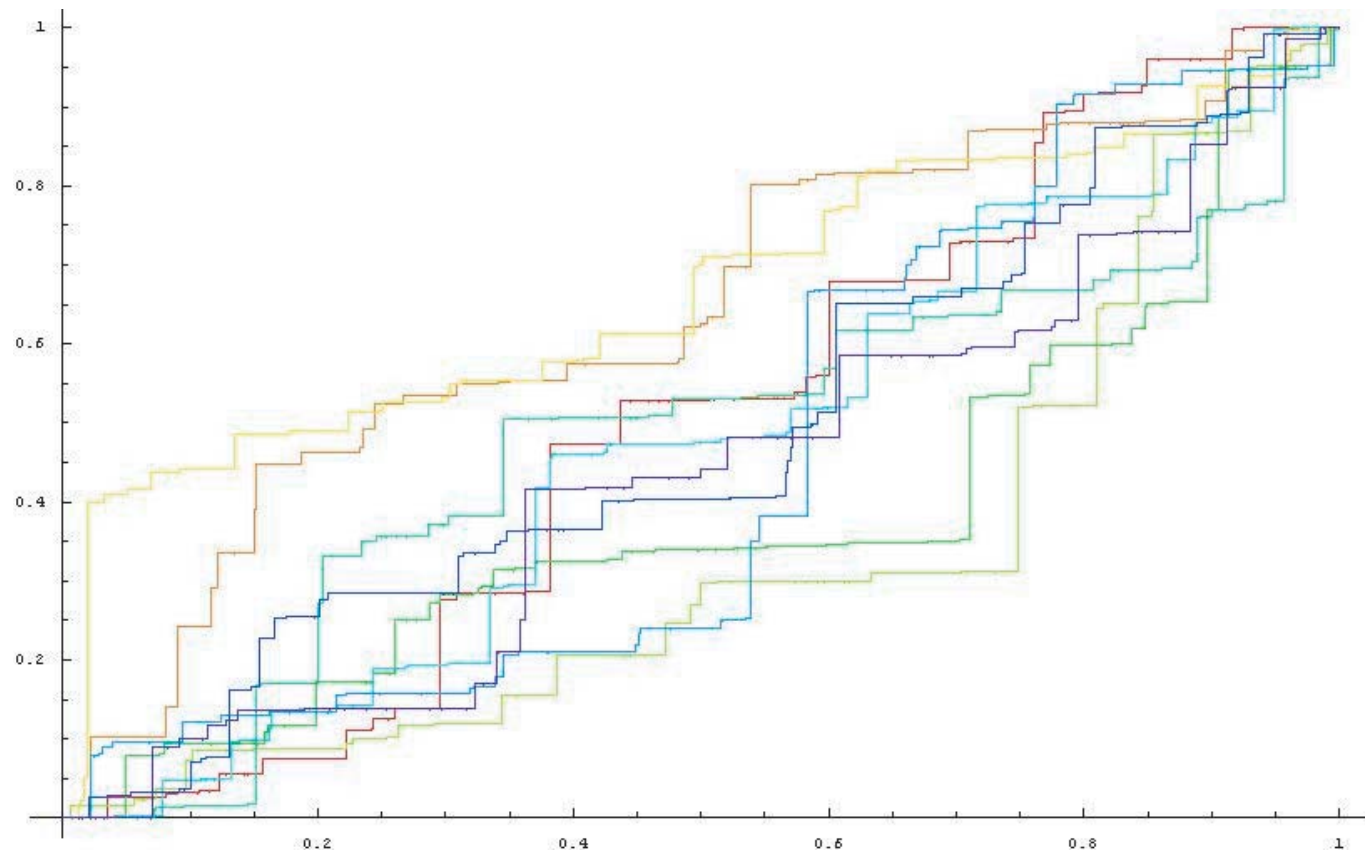where $\mathrm{B}[a,b] = \Gamma[a]\Gamma[b]/\Gamma[a+b]$.

The gamma bridge has the following remarkable property.

For all $T \ge t \ge 0$ the random variables γ_t/γ_T and γ_T are independent.

Sample paths for the gamma bridge

Gamma information and the valuation of contingent claims

Our objective now is to calculate the value at time t of the claim that pays X_T at time T, where X_T is a positive random variable.

We assume that the market filtration is generated by an information process $\{\xi_t\}_{0 \leq t \leq T}$ of the form

$$\xi_t = X_T \gamma_{tT}, \tag{12}$$

where $\{\gamma_{tT}\}$ is a $\mathbb{Q}$-gamma bridge with parameter m.

The gamma bridge $\{\gamma_{tT}\}$ is independent of the random variable X_T, and represents in some sense the noise that obscures the true value of X_T.

The gamma information process $\{\xi_t\}_{0 \leq t \leq T}$ has the Markov property.

The value S_t of the claim at time t is then given by

$$S_t = P_{tT} \mathbb{E}\left[X_T \,\middle|\, \xi_t\right]. \tag{13}$$

The conditional probability density for X_T,

$$\pi_t(x) = \frac{\mathrm{d}}{\mathrm{d}x} \mathbb{Q}\left[X_T \leq x \,\middle|\, \xi_t\right], \tag{14}$$

can be computed by use of the Bayes formula:

$$\pi_t(x) = \frac{p(x)\rho\left(\xi_t \mid X_T = x\right)}{\int_0^\infty p(x)\rho(\xi_t \mid X_T = x)\mathrm{d}x}, \tag{15}$$

where $\{p(x)\}_{0<x<\infty}$ is the *a priori* density function for X_T.

The result is given by

$$\pi_t(x) = \mathbb{1}_{\{x>\xi_t\}}\frac{p(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}}{\int_{\xi_t}^\infty p(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}\mathrm{d}x}. \tag{16}$$

It follows that the conditional expectation can be calculated to give us:

$$S_t = P_{tT}\frac{\int_{\xi_t}^\infty p(x)x^{2-mT}(x-\xi_t)^{m(T-t)-1}\,\mathrm{d}x}{\int_{\xi_t}^\infty p(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}\,\mathrm{d}x}. \tag{17}$$

When X_T is a discrete random variable taking values $\{x_i\}$ with *a priori* probabilities $\{p_i\}$, we have

$$S_t = P_{tT}\frac{\sum_i p_i x_i^{2-mT}(x_i-\xi_t)^{m(T-t)-1}\mathbb{1}_{\{\xi_t<x_i\}}}{\sum_i p_i x_i^{1-mT}(x_i-\xi_t)^{m(T-t)-1}\mathbb{1}_{\{\xi_t<x_i\}}}. \tag{18}$$
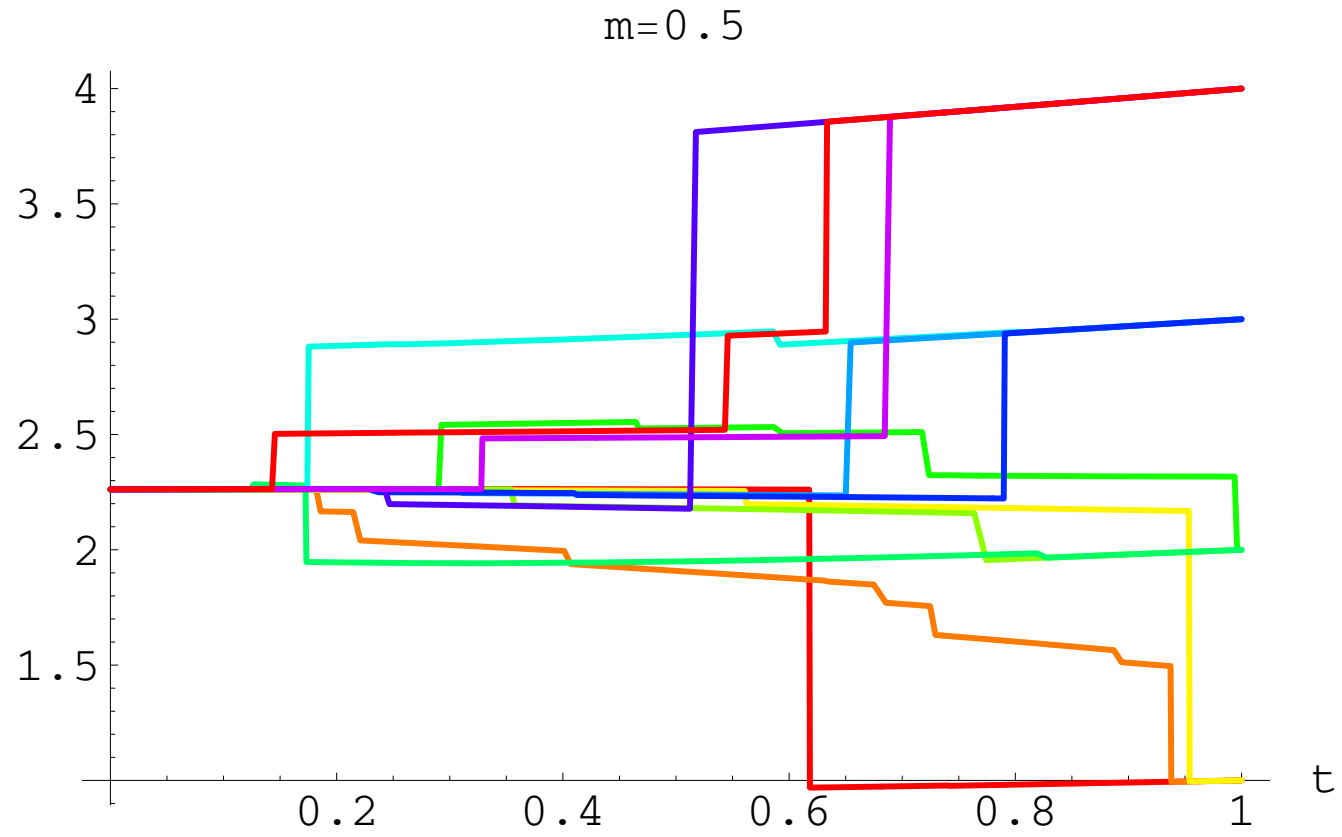
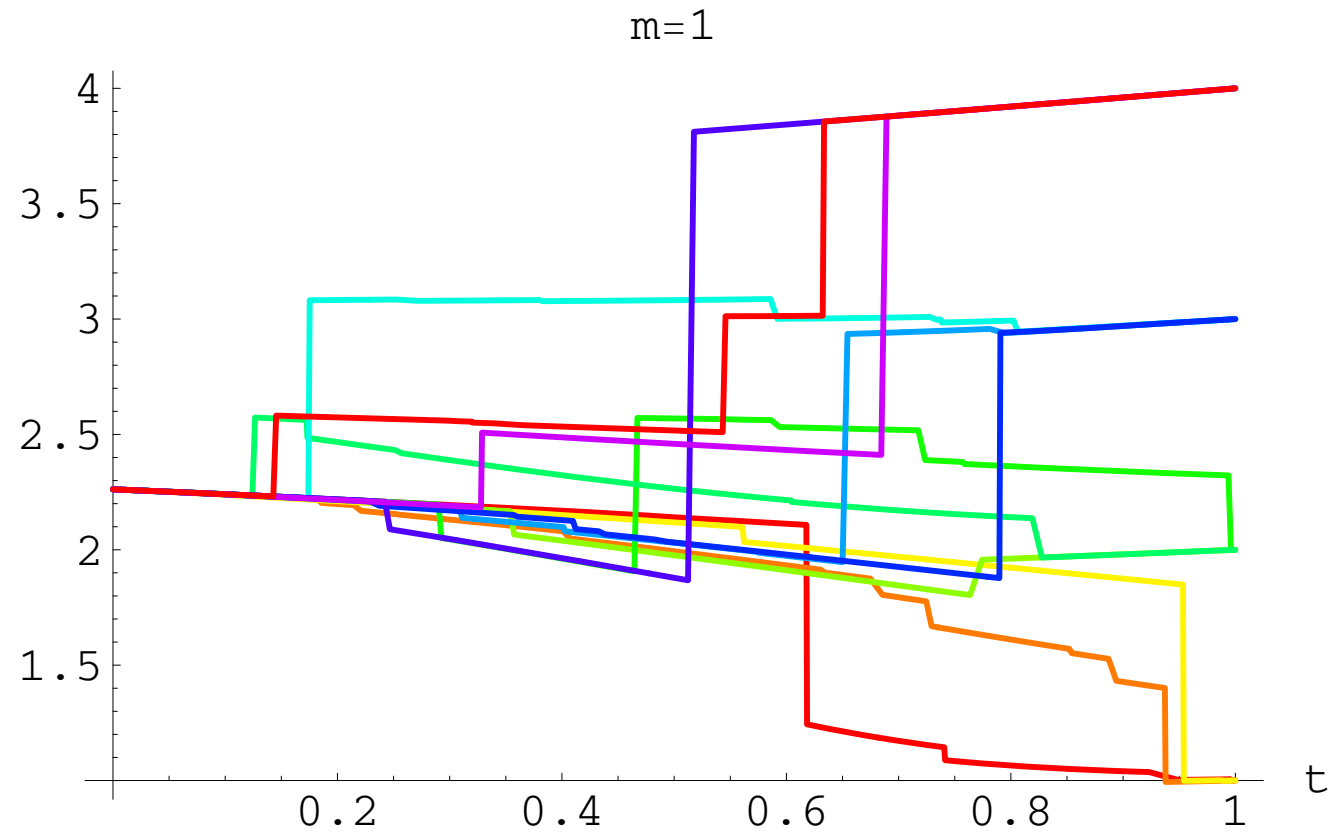Figure 1: Sample paths for the asset price process. The rate parameter is $m = 0.5$. Interest rate is $r = 5\%$.

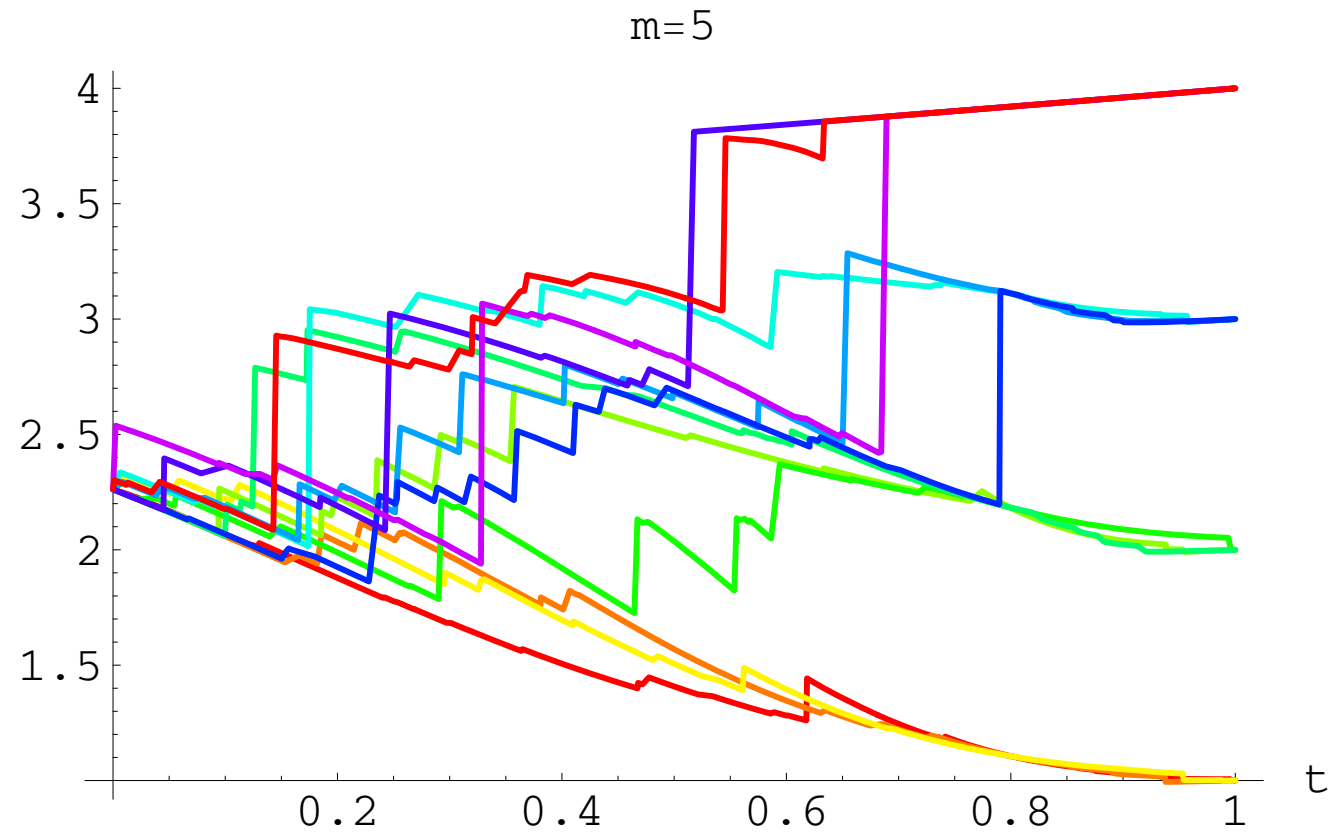Figure 2: Sample paths for the asset price process. The rate parameter is $m = 1.0$. Interest rate is $r = 5\%$.

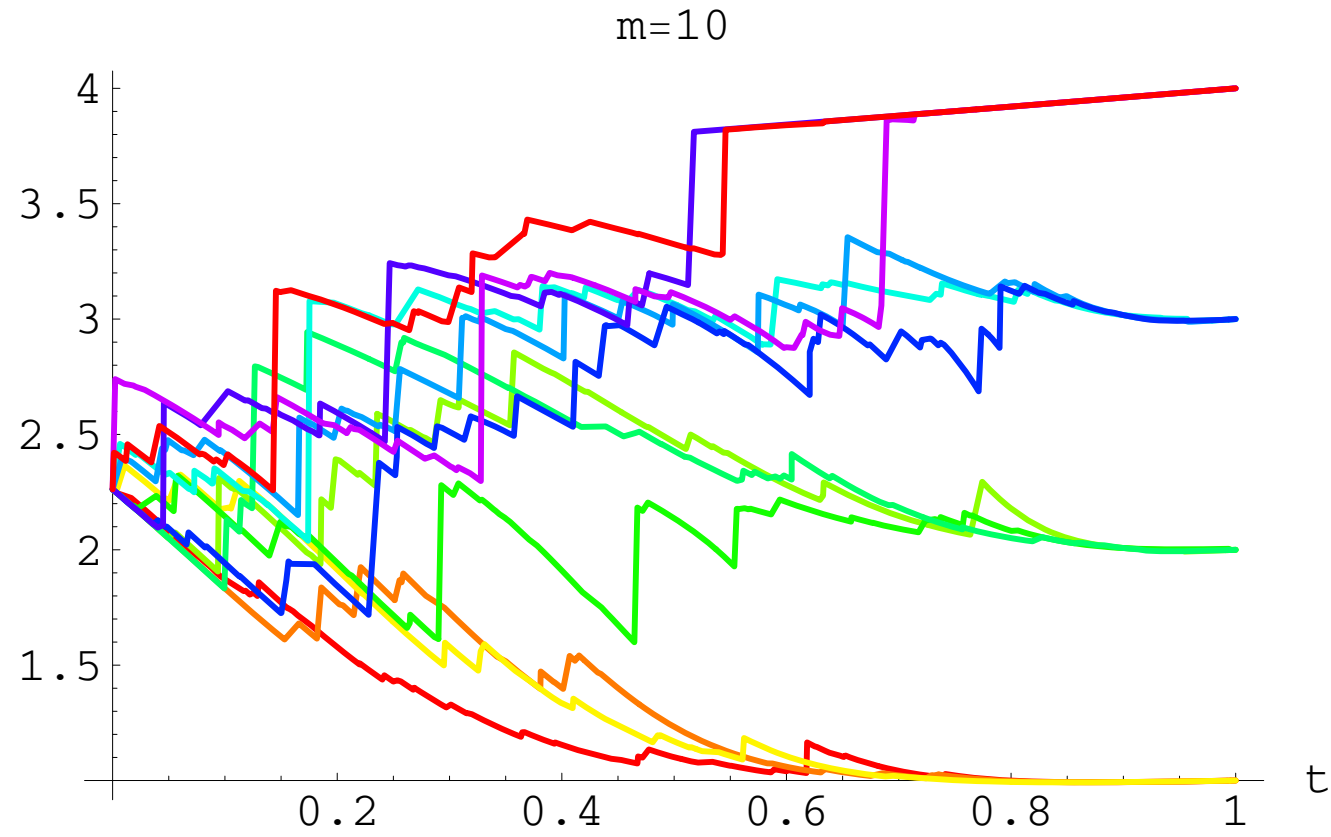Figure 3: Sample paths for the asset price process. The rate parameter is $m = 5.0$. Interest rate is $r = 5\%$.

Figure 4: Sample paths for the asset price process. The rate parameter is $m = 10$. Interest rate is $r = 5\%$.

Valuation of reinsurance contracts

We consider now the valuation of call options on the terminal payout of a aggregate claims process.

Such options have the interpretation of stop-loss reinsurance contracts.

The option maturity is set at time $t < T$, and the strike at K.

If we define the function $S(t, y)$ for $0 \le t \le T$ and $y \ge 0$ by

$$S(t, y) = P_{tT} \frac{\int_y^\infty p(x) x^{2-mT} (x - y)^{m(T-t)-1} \mathrm{d}x}{\int_y^\infty p(x) x^{1-mT} (x - y)^{m(T-t)-1} \mathrm{d}x}, \tag{19}$$

then the price at time t of the asset is given by $S(t, \xi_t)$.

Therefore, the initial value of the call option is

$$
\begin{aligned}
C_0 &= P_{0t} \mathbb{E}\left[(S(t, \xi_t) - K)^+ \right] \\
&= P_{0t} \mathbb{E}\left[\int_0^\infty \delta(\xi_t - y)\, (S(t, y) - K)^+ \, \mathrm{d}y \right] \\
&= \int_0^\infty A_{0t}(y)\, (S(t, y) - K)^+ \, \mathrm{d}y.
\end{aligned}
\tag{20}
$$

Here we have defined

$$A_{0t}(y) = P_{0t}\mathbb{E}[\delta(\xi_t - y)] \tag{21}$$

for the Arrow-Debreu security on the aggregate gains process.

A short calculation shows that

$$A_{0t}(y) = P_{0t}\frac{y^{mt-1}}{\mathrm{B}[mt, m(T-t)]}\int_y^\infty p(x)\, x^{1-mT}(x-y)^{m(T-t)-1}\mathrm{d}x. \tag{22}$$

In terms of $A_{0t}(y)$ the initial call price is given by

$$C_0 = \int_0^\infty A_{0t}(y)\,[S(t,y) - K]^+\,\mathrm{d}y. \tag{23}$$

We thus find

$$C_0 = \frac{P_{0t}}{\mathrm{B}[mt, m(T-t)]}\int_0^\infty \left[P_{tT}\int_y^\infty p(x)\, x^{2-mT}y^{mt-1}(x-y)^{m(T-t)-1}\mathrm{d}x \right.$$
$$\left. -K\int_y^\infty p(x)\, x^{1-mT}y^{mt-1}(x-y)^{m(T-t)-1}\mathrm{d}x \right]^+\,\mathrm{d}y. \tag{24}$$

We let y^* denote the critical value of y such that the argument of the max-function vanishes.

Then for the call price we find

$$C_0 = P_{0t} \int_{y^*}^{\infty} p(x) \, (x \, P_{tT} - K) \, \mathcal{B}(y^*/x) \, \mathrm{d}x, \qquad (25)$$

where

$$\mathcal{B}(u) = \frac{\int_u^1 z^{mt-1} (1-z)^{m(T-t)-1} \mathrm{d}z}{\int_0^1 z^{mt-1} (1-z)^{m(T-t)-1} \mathrm{d}z} \qquad (26)$$

is the complementary beta distribution function.

Note: the option price process $\{C_s\}$ can also be calculated analogously to yield

$$C_s = P_{st} \int_{x=y*}^{\infty} \pi_s(x)(x P_{tT} - K) \, \mathcal{B}\left(\frac{y^* - \xi_s}{x - \xi_s}\right) \mathrm{d}x. \qquad (27)$$

Options on assets having discrete cash flows

When the random variable X_T takes discrete values $\{x_i\}$ with probability $\{p_i\}$ the expression for the call option is given as follows:

$$C_0 = P_{0\tau} \sum_{i=0}^{n} p_i \left(x_i P_{\tau T} - K \right) \mathcal{B}(y^*/x_i) \mathbb{1}_{\{x_i > y^*\}}. \tag{28}$$

In particular, if X_T is a binary variable, this reduces to

$$C_0 = P_{0\tau} \left[p_0 \left(x_0 P_{\tau T} - K \right) \mathcal{B}\left(\frac{y^*}{x_0} \right) + p_1 \left(x_1 P_{\tau T} - K \right) \mathcal{B}\left(\frac{y^*}{x_1} \right) \right], \tag{29}$$

where

$$y^* = \frac{\theta x_1 - x_0}{\theta - 1} \tag{30}$$

and

$$\theta = \left[\frac{p_1(K - P_{\tau T} x_1)}{p_0(P_{\tau T} x_0 - K)} \left(\frac{x_1}{x_0} \right)^{1 - mT} \right]^{\frac{1}{m(T - \tau) - 1}}. \tag{31}$$

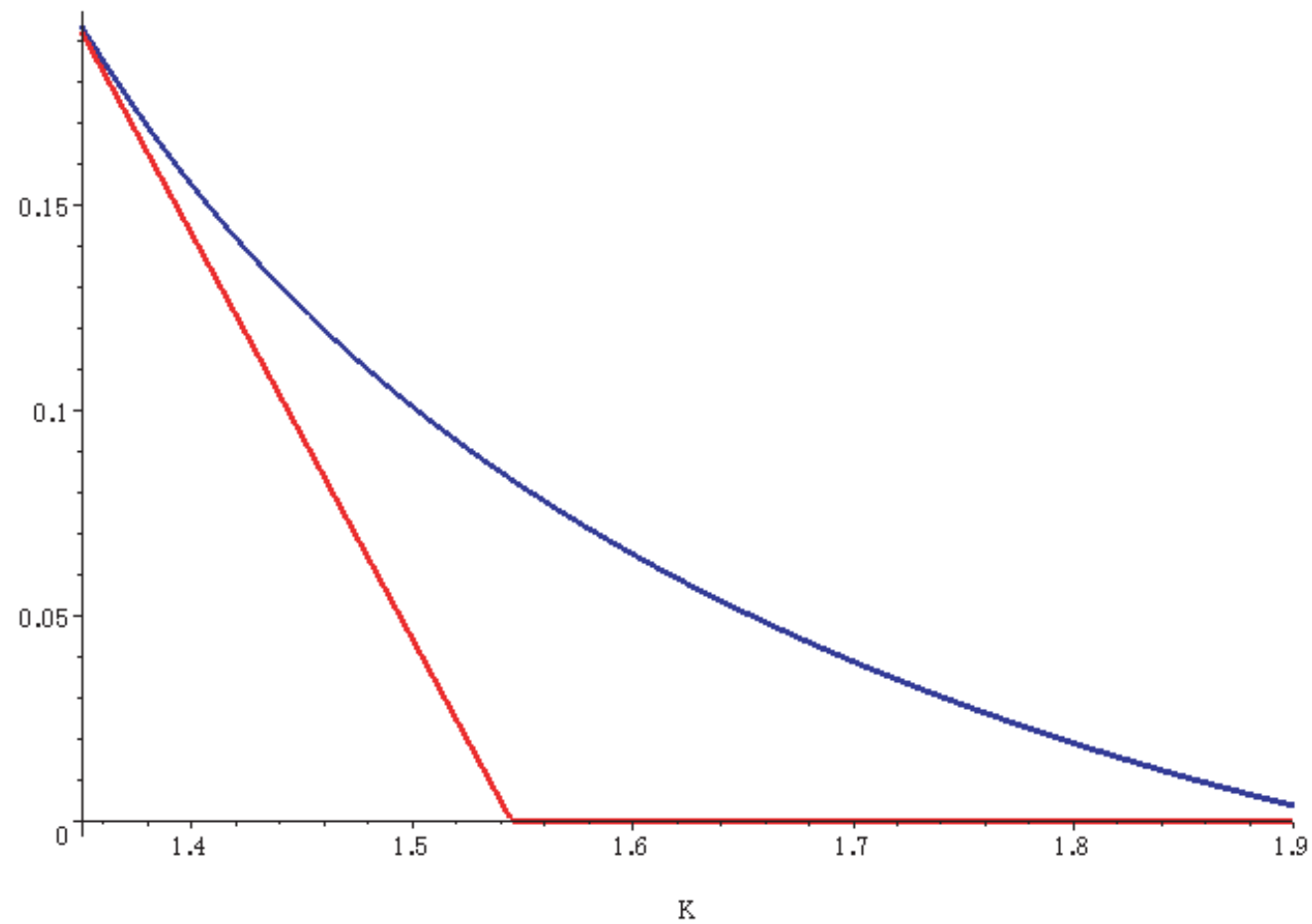Figure 5: Call price as a function of the strike K. The parameters are: $r = 5\%$, $m = 4.5$, $T = 1$ year, $\tau = 0.3$ year, and $S_0 = 1.52196$.
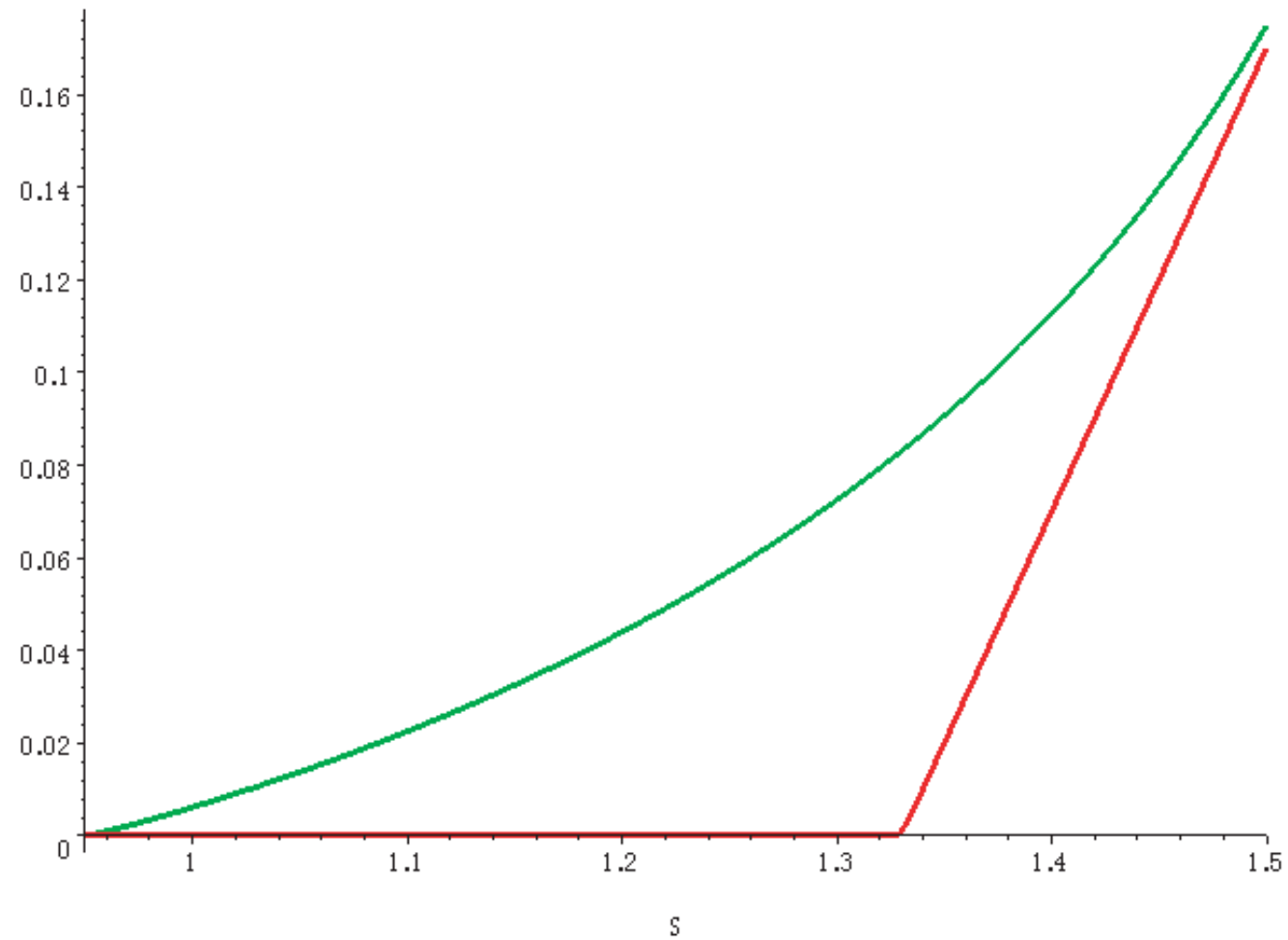
Figure 6: Call price as a function of initial price S_0. The parameters are: $r = 5\%$, $m = 4.5$, $T = 1$ year, $\tau = 0.3$ year, and $K = 1.35$.

Gamma-distributed cash flow

When the random variable X_T is itself gamma distributed with parameter mT, the resulting value process $\{S_t\}$ admits a particularly simple structure.

Specifically, we consider the case for which the *a priori* density function $p(x)$ for X_T is given by

$$p(x) = \frac{1}{\Gamma[mT]} X_0^{-mT} x^{mT-1} \mathrm{e}^{-x/X_0}, \tag{32}$$

where $X_0 > 0$ is a fixed parameter.

For the asset price process we find that

$$S_t = P_{tT}\big(\xi_t + X_0 m(T - t)\big). \tag{33}$$

Therefore, the value process $\{S_t\}$ in this example is a linear function of the cumulative gains process $\{\xi_t\}$.

The price process for the Arrow-Debreu security on ξ_t is

$$A_{st}(y) = P_{st} \frac{X_0^{-m(t-s)}}{\Gamma[m(t-s)]} (y - \xi_s)^{m(t-s)-1} \mathrm{e}^{-(y-\xi_s)/X_0}. \tag{34}$$

It follows that an insurance derivative with payout $(S_t - K)^+$ has the price process

$$C_s = P_{sT} \left[\frac{\Gamma[m(t-s)+1, \tilde{K}_s]}{\Gamma[m(t-s)]} - \tilde{K}_s \frac{\Gamma[m(t-s), \tilde{K}_s]}{\Gamma[m(t-s)]} \right], \qquad (35)$$

where we have defined

$$\tilde{K}_s = P_{tT}^{-1} K / X_0 - \xi_s / X_0 - m(T - t) \qquad (36)$$

and

$$\Gamma[a, z] = \int_z^\infty u^{a-1} \mathrm{e}^{-u} \mathrm{d}u \qquad (37)$$

denotes the incomplete gamma integral.

The initial call price is

$$C_0 = P_{0t} \left[P_{tT} \frac{\Gamma[mt+1, \tilde{K}]}{\Gamma[mt]} - \tilde{K} \frac{\Gamma[mt, \tilde{K}]}{\Gamma[mt]} \right], \qquad (38)$$

where $\tilde{K} = P_{tT}^{-1} K / X_0 - mT$