Matrix Subordinators and Multivariate OU-based Volatility Models

Ole E. Barndorff-Nielsen

Thiele Centre
Department of Mathematical Sciences
University of Aarhus

▪

Based mainly on joint papers with:

Victor Perez-Abreu *CIMAT*
Robert Stelzer *TUM*

Synopsis

- Intro
- Volatility and OU processes
- Matrix subordinators
- Infinite divisibility in cones
- CLT for RMPV
- Positive definite matrix processes of OU type
- Roots of positive definite processes

Intro

Let Y_t denote a d-dimensional vector of log prices, modelled as a Brownian semimartingale

$$Y_t = \int_0^t a_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s$$

- ⋆ OU modelling of $\Sigma = \sigma^\top \sigma$. One-dimensional case: realism and analytical tractability
- ⋆ *Multipower Variation* RMPV: Basis for inference on $\Sigma_t^+ = \int_0^t \Sigma_s \mathrm{d}s$ where $\Sigma_s = \sigma_s^\top \sigma_s$ and more generally on $\Sigma_t^{+r} = \int_0^t \Sigma_s^r \mathrm{d}s$.
- ⋆ The MPV theory uses SDE representations of $\mathrm{d}\sigma$ (not $\mathrm{d}\Sigma$). Need SDE representations of Σ^r, in particular $\Sigma^{1/2}$

Volatility and OU processes

Univariate OU volatility

$$d\sigma_t^2 = -\lambda\sigma_{t-}^2 dt + dL_{\lambda t}$$

where $\lambda > 0$ is a parameter and L is a *subordinator*, i.e. a Lévy process with nonnegative increments.

Volatility and OU processes

The solution can be shown to be

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)} dL_{s\lambda}$$

Provided $E(\log^+(L_t)) < \infty$ there is a *unique stationary solution* given by

$$\sigma_t^2 = \int_{-\infty}^t e^{-\lambda(t-s)} dL_{\lambda s}$$

Volatility and OU processes

There is a vast literature concerning the extension of OU processes to $\mathbb{R}^d$-valued processes.

By identifying M_d, the class of $d \times d$ matrices, with $\mathbb{R}^{d^2}$ one immediately obtains matrix valued processes.

So for a given Lévy process $(L_t)_{t\in\mathbb{R}}$ with values in M_d and a linear operator $\mathbf{A} : M_d \to M_d$, a solution to the SDE

$$dX_t = \mathbf{A}X_{t-}dt + dL_t$$

is termed a *matrix-valued process of Ornstein-Uhlenbeck type*.

Volatility and OU processes

As in the univariate case one can show that for some given initial value X_0 the solution is unique and given by

$$X_t = e^{\mathbf{A}t} X_0 + \int_0^t e^{\mathbf{A}(t-s)} dL_s.$$

Provided $E(\log^+ \|L_t\|) < \infty$ and $\sigma(\mathbf{A}) \in (-\infty, 0) + i\mathbb{R}$, there exists a unique stationary solution given by

$$X_t = \int_{-\infty}^t e^{\mathbf{A}(t-s)} dL_s.$$

Matrix subordinators

However, in order to obtain positive semidefinite Ornstein-Uhlenbeck processes we need to consider *matrix subordinators* as driving Lévy processes.

Let $\bar{S}_d^+$ be the closure of the cone S_d^+ of positive definite matrices in M_d.

Definition A process L with values in $\bar{S}_d^+$ and having independent stationary increments is called a *matrix subordinator*

Infinite divisibility in the cone $\bar{S}_d^+$

A random matrix M is *infinitely divisible in* $\bar{S}_d^+$ if and only if for each integer $p \geq 1$ there exist p independent identically distributed random matrices $M_1, ..., M_p$ in $\bar{S}_d^+$ such that $M \overset{law}{=} M_1 + ... + M_p$.

Lévy-Khintchine representation (Skorohod (1991))
A random matrix $M \in \bar{S}_d^+$ is infinitely divisible in $\bar{S}_d^+$ if and only if its cumulant transform is of the form

$$\mathcal{C}(\Theta; M) = i\mathrm{tr}(\gamma\Theta) + \int_{\bar{S}_d^+} (e^{i\mathrm{tr}(X\Theta)} - 1)\rho(\mathrm{d}X), \quad \Theta \in S_d^+,$$

where $\gamma \in \bar{S}_d^+$ is called the drift and the Lévy measure ρ is such that $\rho(S_d^+ \backslash \bar{S}_d^+) = 0$ and ρ has order of singularity

$$\int_{\bar{S}_d^+} \min(1, \mathrm{tr}(X))\rho(\mathrm{d}X) < \infty.$$

Infinite divisibility in the cone $\bar{S}_d^+$

Lévy-Ito decomposition:

If $\{L_t\}$ is a matrix subordinator with the above Lévy-Khintchine representation then it has a Lévy-Itô decomposition

$$L_t = t\gamma + \int_0^t \int_{\mathbb{S}_d^+ \setminus \{0\}} x\mu(ds, dx)$$

where $\gamma \in \bar{S}_d^+$ is a deterministic drift and $\mu(ds, dx)$ a Poisson random measure on $\mathbb{R}^+ \times \bar{S}_d^+$ with

$$E(\mu(ds, dx)) = Leb(ds)\nu(dx),$$

Leb denoting the Lebesgue measure and ν the Lévy measure of L_t.

Examples

⋆ *Quadratic Covariation* of d-dimensional Lévy processes

⋆ *Gamma type matrix distribution* Lévy density:

$$\frac{|\Sigma|^{-<d>}}{\left(\mathrm{tr}\left(X\Sigma^{-1}\right)\right)^{[d]}} e^{\mathrm{tr}\left(-X\Sigma^{-1}\right)}$$

where $< d >= (d+1)/2$ and $[d] = (d+1)\,d/2$.

Kumulant transform:

$$\mathcal{K}(\Theta, R) = \int_{\bar{S}_d^+} \log(1 + \mathrm{tr}(U\Sigma^{1/2}\Theta\Sigma^{1/2}))^{-1}\mathrm{d}U.$$

Examples

⋆ *Bessel matrix distribution* Lévy density:

$$|\Sigma|^{-<d>} \int_{\mathrm{Y}>0} \mathrm{etr}(-\left\{\mathbf{X}\mathrm{Y}^{-1} + \Sigma^{-1}\mathbf{Y})\right\} \left(\mathrm{tr}(\mathbf{Y}\Sigma^{-1})\right)^{-[d]-\beta} \frac{\mathrm{dY}}{|\mathrm{Y}|^{<d>}}.$$

where $\mathbf{X}$ and Y are the *anti-matrices* of X and Y.

Interlude: CLT for RMPV

Central Limit Theory for Realised Multipower Variation
(B-N, Jacod, Graversen, Podolskij and Shephard (2006))

Recall: For a wide class of real–valued processes Y, including all semi-martingales, the *realised quadratic variation process*

$$V(Y;2)_t^n = \sum_{i=1}^{[nt]} (Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}})^2$$

converges in probability, as $n \to \infty$ and for all $t \geq 0$, towards the quadratic variation process $V(Y;2)_t$ (usually denoted by $[Y, Y]_t$).

Interlude: CLT for RMPV

Next, let r, s be nonnegative numbers. The *realised bipower variation process* of order (r, s) is the increasing processes defined as:

$$V(Y; r, s)^n_t = n^{\frac{r+s}{2}-1} \sum_{i=1}^{[nt]} |Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}|^r \, |Y_{\frac{i+1}{n}} - Y_{\frac{i}{n}}|^s.$$

Clearly $V(Y; 2)^n = V(Y; 2, 0)^n$.

The bipower variation process of order (r, s) for Y, denoted by $V(Y; r, s)_t$, is the limit in probability, if it exists for all $t \geq 0$, of $V(Y; r, s)^n_t$.

Uses: Testing for jumps; Estimation of $\int_0^t \sigma_s^4 \mathrm{d}s$ in the presence of jumps; ...

Interlude: CLT for RMPV

Extension to the multidimensional case.

Now $Y = (Y^j)_{1 \le j \le d}$ is taken as d–dimensional.

The *realised cross–multipower variation processes* are defined by

$$V(Y^{j_1}, \ldots, Y^{j_N}; r_1, \ldots, r_N)_t^n = n^{\frac{r_1 + \ldots + r_N}{2} - 1} \sum_{i=1}^{[nt]} |Y^{j_1}_{\frac{i}{n}} - Y^{j_1}_{\frac{i-1}{n}}|^{r_1} \ldots |Y^{j_N}_{\frac{i+N-1}{n}} - Y^{j_N}_{\frac{i+N-2}{n}}|^{r_N}.$$

Interlude: CLT for RMPV

More generally still, let

$$X^n(g,h)_t = \frac{1}{n} \sum_{i=1}^{[nt]} g(\sqrt{n}\,\Delta_i^n Y)h(\sqrt{n}\,\Delta_{i+1}^n Y)$$

where $\Delta_i^n Y = Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}$, g and h are two maps on R^d, taking vakues in $\mathcal{M}_{d_1,d_2}$ and $\mathcal{M}_{d_2,d_3}$ respectively. So $X^n(g,h)_t$ takes its values in $\mathcal{M}_{d_1,d_3}$.

We refer to $X^n(g,h)$ as the realised multipower variation (RMPV) associated to g and h.

Interlude: CLT for RMPV

To derive a CLT for RMPV we need the following structural assumptions:

Hypothesis (H): We have

$$Y_t = Y_0 + \int_0^t a_s ds + \int_0^t \sigma_{s-} \, dW_s,$$

where W is a standard d'–dimensional BM, a is predictable R^d–valued locally bounded, and σ is $\mathcal{M}_{d,d'}$–valued càdlàg with $\Sigma = \sigma\sigma^\top$ invertible.

Interlude: CLT for RMPV

Hypothesis (H'): We have

$$\begin{aligned}\sigma_t \; &= \; \sigma_0 + \int_0^t a'_s ds + \int_0^t \sigma'_{s-} dW_s + \int_0^t v_{s-} dV_s \\ &+ \int_0^t \int_E \varphi \circ w(s-,x)(\mu - \nu)(ds,dx) \\ &+ \int_0^t \int_E (w - \varphi \circ w)(s-,x)\mu(ds,dx).\end{aligned}$$

where ****

Interlude: CLT for RMPV

Hypothesis (K): The function g and h are even and continuously differentiable, with partial derivatives having at most polynomial growth.

Now, recall that

$$X^n(g,h)_t = \frac{1}{n} \sum_{i=1}^{[nt]} g(\sqrt{n}\, \Delta_i^n \mathrm{Y}) h(\sqrt{n}\, \Delta_{i+1}^n \mathrm{Y})$$

Under **(H)**, **(H')** and **(K)**, $X^n(g,h)$ converges in probability to a process $X(g,h)$.

Interlude: CLT for RMPV

Theorem CLT for RMPV Under **(H)**, **(H')** and **(K)** the process

$$\sqrt{n}\left(X^{n}(g,h)-X(g,h)\right)$$

converges *stably in law* to the limiting process $U(g,h)$ given componentwise by

$$U(g,h)_{t}^{jk}=\sum_{j'=1}^{d_1}\sum_{k'=1}^{d_3}\int_0^t \alpha\left(\sigma_s,g,h\right)^{jk,j'k'}\,\mathrm{d}W_s'^{j'k'}$$

where W' is a multidimensional Brownian motion, independent of all the previous random objects, and where the coefficients $\alpha\left(\sigma_s,g,h\right)$ satisfy ****.

Positive semidefinite matrix processes of OU type

$$dX_t = \mathbf{A}X_{t-}dt + dL_t$$

Proposition Let L_t be a matrix subordinator, assume that the linear operator $\mathbf{A}$ satisfies $\exp(\mathbf{A}t)(\bar{S}_d^+) \subseteq \bar{S}_d^+$ for all $t \in \mathbb{R}^+$ and let $X_0 \in \bar{S}_d^+$.

Then the Ornstein-Uhlenbeck process $(X_t)_{t\in\mathbb{R}^+}$ satisfying $dX_t = \mathbf{A}X_{t-}dt + dL_t$ with initial value X_0 takes only values in $\bar{S}_d^+$.

If $E(\log^+ \|L_t\|) < \infty$ and $\sigma(\mathbf{A}) \in (-\infty, 0) + i\mathbb{R}$, then the unique stationary solution $(X_t)_{t\in\mathbb{R}}$ takes values in $\bar{S}_d^+$ only.

Positive semidefinite matrix processes of OU type

Which linear operators $\mathbf{A}$ can one actually take to obtain both a unique stationary solution and ensure positive semidefiniteness?

The condition $\exp(\mathbf{A}t)(\mathbb{S}_d^+) \subseteq \mathbb{S}_d^+$ means that for all $t \in \mathbb{R}^+$ the exponential operator $\exp(\mathbf{A}t)$ has to preserve positive definiteness. So one needs to know first which linear operators on S_d^+ preserve positive definiteness.

Positive semidefinite matrix processes of OU type

- ⋆ Let $\mathbf{A} : S_d \longrightarrow S_d$ be a linear operator. Then $\mathbf{A}(\bar{S}_d^+) = \bar{S}_d^+$, if and only if there exists a matrix $B \in GL_d$ such that $\mathbf{A}$ can be represented as $X \mapsto BXB^*$.

- ⋆ Assume the operator $\mathbf{A} : \bar{S}_d^+ \longrightarrow \bar{S}_d^+$ is representable as $X \mapsto AX + XA^*$ for some $A \in M_d$. Then $e^{\mathbf{A}t}$ has the representation $X \mapsto e^{At} X e^{A^* t}$ and $e^{\mathbf{A}t}(\bar{S}_d^+) = \bar{S}_d^+$ for all $t \in \mathbb{R}$.

Positive semidefinite matrix processes of OU type

For a linear operator $\mathbf{A}$ of the latter type (i.e. $X \mapsto AX + XA^*$) the SDE for the OU process becomes

$$dX_t = (AX_{t-} + X_{t-}A^*)dt + dL_t$$

and the solution is

$$X_t = e^{At}X_0e^{A^*t} + \int_0^t e^{A(t-s)}dL_s e^{A^*(t-s)}.$$

Positive semidefinite matrix processes of OU type

Theorem Let $(L_t)_{t\in\mathbb{R}}$ be a matrix subordinator with $E(\log^+ \|L_t\|) < \infty$ and let $A \in M_d$ such that $\sigma(A) \subset (-\infty, 0) + i\mathbb{R}$.

Then the stochastic differential equation of Ornstein-Uhlenbeck type

$$dX_t = (AX_{t-} + X_{t-}A^*)dt + dL_t$$

has a unique *stationary* solution

$$X_t = \int_{-\infty}^{t} e^{A(t-s)} dL_s e^{A^*(t-s)}.$$

Moreover, $X_t \in \bar{S}_d^+$ for all $t \in \mathbb{R}$.

Positive semidefinite matrix processes of OU type

Conditions ensuring that the stationary OU type process X_t is almost surely strictly positive definite can be obtained:

Theorem If $\gamma \in S_d^+$ or $\nu(S_d^+) > 0$, then the stationary distribution P_X of X_t is concentrated on S_d^+.

Positive semidefinite matrix processes of OU type

Extensive recent work by Christian Pigorsch, LMU, jointly with Robert Stelzer, TUM, on properties, extensions and applications of this general multivariate SV-OU framework.

Roots of positive semidefinite processes

To discuss the root questions we need a suitable *Itô formulae for finite variation processes in open sets*

Definition Local Boundedness Let $(V, \|\cdot\|_V)$ be either $\mathbb{R}^d, S_d^+$ or S_d with $d \in \mathbb{N}$ and equipped with the norm $\|\cdot\|_V$, let $a \in V$ and let $(X_t)_{t\in\mathbb{R}^+}$ be a V-valued stochastic process. We say that X_t is *locally bounded away from* a if there exists a sequence of stopping times $(T_n)_{n\in\mathbb{N}}$ increasing to infinity almost surely and a real sequence $(d_n)_{n\in\mathbb{N}}$ with $d_n > 0$ for all $n \in \mathbb{N}$ such that $\|X_t - a\|_V \geq d_n$ for all $0 \leq t < T_n$.

Likewise, we say for some open set $C \in V$ that the process $X_t$is *locally bounded within* C if there exists a sequence of stopping times $(T_n)_{n\in N}$ increasing to infinity almost surely and a sequence of compact convex subsets $D_n \subset C$ with $D_n \subset D_{n+1}$forall $n \in N$ such that $X_t \in D_n$ for all $0 \leq t < T_n$.

Roots of positive semidefinite processes

Proposition *Itô formulae for finite variation processes in open sets* Let $(X_t)_{t\in\mathbb{R}^+}$ be a cadlag $\mathbb{R}^d$-valued process of finite variation (thus a semimartingale) with associated jump measure μ_X on $\left(\mathbb{R}^+ \times \mathbb{R}^d\backslash\{0\}, \mathcal{B}\left(\mathbb{R}^+ \times \mathbb{R}^d\backslash\{0\}\right)\right)$ and let $f : C \rightarrow \mathbb{R}^m$ be continuously differentiable, where $C \subseteq \mathbb{R}^d$ is an open set. Assume that the process $(X_t)_{t\in\mathbb{R}^+}$ is *locally bounded within* C. Then:

Roots of positive semidefinite processes

the process X_t as well as its left limit process X_{t-} take values in C at all times $t \in \mathbb{R}^+$ and

$$\begin{aligned} f(X_t) &= f(X_0) + \int_0^t Df(X_{s-})dX_s^c \\ &\quad + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} (f(X_{s-} + x) - f(X_{s-}))\mu_X(ds, dx). \end{aligned}$$

Roots of positive semidefinite processes

Univariate case

Theorem Let $(X_t)_{t\in\mathbb{R}^+}$ be a given adapted cadlag process which takes values in $\mathbb{R}^+\backslash\{0\}$, is locally bounded away from zero and can be represented as

$$dX_t = c_t dt + \int_{\mathbb{R}^+\backslash\{0\}} g(t-,x)\mu(dt,dx)$$

where c_t is a predictable and locally bounded process, μ a Poisson random measure on $\mathbb{R}^+ \times \mathbb{R}^+\backslash\{0\}$ and $g(s,x)$ is $\mathcal{F}_s \times \mathcal{B}(\mathbb{R}^+\backslash\{0\})$ measurable in (ω, x) and cadlag in s. Moreover, $g(s,x)$ takes only non-negative values. Then:

Roots of positive semidefinite processes

for any $0 < r < 1$ the unique positive process $Y_t = X_t^r$ is representable as

$$Y_0 = X_0^r, \quad dY_t = a_t dt + \int_{\mathbb{R}^+ \setminus \{0\}} w(t-, x) \mu(dt, dx),$$

where the drift

$$a_t := r X_{t-}^{r-1} c_t$$

is predictable and locally bounded and where

$$w(s, x) := (X_s + g(s, x))^r - (X_s)^r$$

is $\mathcal{F}_s \times \mathcal{B}(\mathbb{R}^+)$ measurable in (ω, x) and cadlag in s. Moreover, $w(s, x)$ takes only non-negative values.

Roots of positive semidefinite processes

When applied to subordinators this gives

Corollary Let $(L_t)_{t\in\mathbb{R}^+}$ be a Lévy subordinator with initial value $L_0 \in \mathbb{R}^+$, associated drift γ and jump measure μ. Then for $0 < r < 1$ we have that the unique positive process L_t^r is of finite variation and

$$dL_t^r = r\gamma L_{t-}^{r-1}dt + \int_{\mathbb{R}^+\setminus\{0\}} \left((L_{t-}+x)^r - L_{t-}^r\right)\mu(dt,dx),$$

where the drift $r\gamma L_{t-}^{r-1}$ is predictable. Moreover, the drift is locally bounded if and only if $L_0 > 0$ or $\gamma = 0$.

Roots of positive semidefinite processes

Multivariate case Generalisation of previous results:

Theorem Let $(X_t)_{t\in\mathbb{R}^+}$ be a given adapted cadlag process which takes values in S_d^+, is locally bounded within S_d^+ and can be represented as

$$dX_t = c_t dt + \int_{\bar{S}_d^+\setminus\{0\}} g(t-,x)\mu(dt,dx)$$

where c_t is an S_d^+-valued, predictable and locally bounded process, μ a Poisson random measure on $\mathbb{R}^+ \times \bar{S}_d^+\setminus\{0\}$, and $g(s,x)$ is $\mathcal{F}_s \times \mathcal{B}(\bar{S}_d^+\setminus\{0\})$ measurable in (ω,x) and cadlag in s. Furthermore, $g(s,x)$ takes only values in $\bar{S}_d^+$. Then

Roots of positive semidefinite processes

the unique positive definite square root process $Y_t = \sqrt{X_t}$ is given by

$$Y_0 = \sqrt{X_0}, dY_t = a_t dt + \int_{\bar{S}_d^+ \setminus \{0\}} w(t-, x)\mu(dt, dx),$$

with

$$a_t = \mathbf{X}_{t-}^{-1} c_t,$$

where $\mathbf{X}_{t-}$ is the linear operator $Z \mapsto \sqrt{X_{t-}} Z + Z\sqrt{X_{t-}}$ on M_d and

$$w(s,x) := \sqrt{X_s + g(s,x)} - \sqrt{X_s}$$

Moreover, $w(s,x)$ takes only positive semidefinite values.

Roots of positive semidefinite processes

Corollary Let $(L_t)_{t\in\mathbb{R}^+}$ be a matrix subordinator with initial value $L_0 \in \bar{S}_d^+$, associated drift γ and jump measure μ. Then the unique positive semidefinite process $\sqrt{L_t}$ is of finite variation and, provided that either $L_0 \in S_d^+$ or $\gamma \in S_d^+ \cup \{0\}$,

$$d\sqrt{L_t} = \mathbb{L}_{t-}^{-1}\gamma dt + \int_{\bar{S}_d^+\setminus\{0\}} \left(\sqrt{L_{t-}+x} - \sqrt{L_{t-}}\right)\mu(dt,dx),$$

where $\mathbb{L}_{t-}$ is the linear operator on M_d with $Z \mapsto \sqrt{L_{t-}}Z + Z\sqrt{L_{t-}}$. The drift $\mathbb{L}_{t-}^{-1}\gamma$ is predictable, and additionally locally bounded provided $L_0 \in \bar{S}_d^+$ or $\gamma = 0$.

Roots of Ornstein-Uhlenbeck processes

Finally we specialise to the behaviour of the roots of positive Ornstein-Uhlenbeck processes.

Recall that the driving Lévy process L_t is assumed to be a (matrix) subordinator.

Univariate case

Let X_t be a stationary process of OU type with driving Lévy subordinator L_t (having non-zero Lévy measure) with a vanishing drift γ. Then for $0 < r < 1$ the stationary process $Y_t = X_t^r$ can be represented as

$$Y_t = \int_{-\infty}^{t} \int_{\mathbb{R}^+ \setminus \{0\}} e^{-\lambda r(t-s)} \left((X_{s-} + x)^r - X_{s-}^r \right) \mu(ds, dx).$$

Roots of Ornstein-Uhlenbeck processes

Multivariate case

Proposition Let X_t be a stationary process of OU type with driving matrix subordinator L_t with a vanishing drift γ. Then the stationary process $Y_t = \sqrt{X_t}$ can be represented as

$$\int_{-\infty}^{t} \int_{\bar{S}_d^+ \setminus \{0\}} \left(\sqrt{e^{A(t-s)}(X_{s-} + x)e^{A^*(t-s)}} - \sqrt{e^{A(t-s)}X_{s-}e^{A^*(t-s)}} \right) \mu(dx, ds)$$

References

[BNGJPS06] Barndorff-Nielsen, O.E., Graversen, S.E., Jacod, J., Podolskij, M. and Shephard, N. (2006): A central limit theorem for realised power and bipower variations of continuous semimartingales. In Yu. Kabanov, R. Liptser and J. Stoyanov (Eds.): *From Stochastic Calculus to Mathematical Finance*. Festschrift in Honour of A.N. Shiryaev. Heidelberg: Springer. N. Pp. 33-68.

[BNPA07] Barndorff-Nielsen, O.E. and Pérez-Abreu, V. (2006): Matrix subordinators and related Upsilon transformations. *Theory of Probability and Its Applications* (To appear).

[BNStel07] Barndorff-Nielsen, O.E. and Stelzer, R. (2007): Positive-definite matrix processes of finite variation. *Probability and Mathematical Statistics* **27**, 3-43.