Contagion Effects and Collateralized Credit Value Adjustments for Credit Default Swaps

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Risk Management of Counterparty Credit Risk

Counterparty Credit Risk (CCR) is concerned with the risk that one of the parties in a OTC derivitave transaction could default before the final settlement of the transactions cash flows.

Tools of counterparty credit risk management:

- Hedging
- Economic Capital
- Collateralization

Content of this talk: Collateralization of OTC-CDS

- Effectivity of different collateralization strategies
- Credit value adjustments
- Impact of contagion effects

Notations

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$ denotes the risk-neutral filtered probability space. We use the abbreviations B, R and S to denote the protection buyer, the reference entity respectively the protection seller. Moreover:

- τ_i default time of entity $i \in \{B, R, S\}$,
- $\tau = \min(\tau_B, \tau_R, \tau_S)$,
- Default indicator processes $H_t^i := 1_{\{\tau_i \leq t\}}$ for $i \in \{B, R, S\}$ and we put $H := (H^B, H^R, H^S)$,
- ξ entity which defaults first,
- LGD_i loss given default of entity i,
- T maturity of the CDS.

Risk-free Credit Default Swap

P_t denotes the time t price of a riskfree CDS, i.e. a CDS without counterparty risk and without collateralization.
We assume that the cashflows arising from a risk free CDS from time t to time s are given by:

$$\Pi(t,s) \quad := \quad 1_{\{t<\tau_R\leq s\}}\,\mathrm{LGD}_R\,D(t,\tau_R) - \int_t^s S_R D(t,u) 1_{\{\tau_R>u\}} du,$$

where S_R represents the spread of the CDS.
Moreover, we define the price of a risk free CDS by $P_t := \mathbb{E}\left(\Pi(t,T)|\mathcal{F}_t\right)$.

Collateralization

Collateralization refers to the practice of posting securities (the so-called collateral) that serve as a pledge for the collateral taker.

Mathematical description of collateralization

A collateralization strategy is an $\mathcal{F}$-adapted RCLL process $(C_t)_{t\in[0,T]}$ with $C_t : \Omega \to \mathbb{R}$. C_t denotes the amount of collateral which is available to the protection buyer ($C_t > 0 \Leftrightarrow B$ is the collateral taker).

Note that this definition implies the following properties:

- The amount of collateral can be updated continously.
- C is updated only wrt current information, for example current price changes of the CDS.

Examples of collateralization strategies

We will only consider collateralization strategies of the form $C_t = g(t, P_t)$ with a deterministic function. Among others the following strategies belong to this class:

- No collateralization: $C \equiv 0$.
- Threshold collateralization: for an initial margin γ and thresholds M_1, $M_2 \geq 0$:

$$C_t^{\gamma, M_1, M_2} := \gamma + (P_t - M_1)1_{\{P_t > M_1\}} + (P_t + M_2)1_{\{P_t < -M_2\}}$$

Credit Value Adjustments

- They are used to take the credit worthiness of the contracting parties into account (in pricing).
- The price of the contingent claim is decomposed in the following way:

$$\begin{aligned} \textit{price} = \ & \text{(counterparty) risk-free price} \\ + \ & \text{adjustment for seller} \\ - \ & \text{adjustment for buyer ,} \end{aligned}$$

  where adjustment for the seller = Credit Value Adjustment (CVA) and adjustment for the buyer = Debt Value Adjustment (DVA).

For more details on the derivation of these adjustments consult [Brigo et al. 2011].

Effectivity of a collateralization strategy

Two reasons why collateralization may result in losses:

- Amount of collateral is not high enough to cover replacement costs.
- Counterparty is unable to return posted collateral (re-hypothecation).

For a given price process P_t of the risk-free CDS and a given strategy C we define (see [Brigo et al. 2011])

$$\begin{aligned}\mathrm{CCVA}(T,C) := \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=S\}}D(0,\tau)(\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ \\ + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+)\big)\\ \mathrm{CDVA}(T,C) := \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=B\}}D(0,\tau)(\mathrm{LGD}_B(C_{\tau-}^- - P_\tau^-)^- \\ + \mathrm{LGD}'_B(P_\tau^+ - C_{\tau-}^+)^-)\big).\end{aligned}$$

as measures for the remaining risk faced by B and S. A good strategy should give low values for $m(C) := \mathrm{CCVA}(T,C) + \mathrm{CDVA}(T,C)$.

Model framework I

We use the model framework from [Frey and Schmidt 2012]:

- X is a finite state Markov chain with state space $S^X = \{1, \dots, K\}$ and generator matrix given by $W = (w_{ij})_{1 \le i,j \le K}$.
- τ_B, τ_R and τ_S are conditionally independent, doubly stochastic random times whose default intensities are increasing functions of X. Therefore

$$\mathbb{Q}(\tau_R > t_1, \tau_B > t_2, \tau_S > t_3 \mid \mathcal{F}_\infty^X) = \prod_{i \in \{B,R,S\}} \exp\Big(- \int_0^{t_i} \lambda_i(X_s) ds \Big)$$

where $\lambda_i : S^X \to \mathbb{R}$.

Model framework II

We consider two models, which differ with respect to the available investor information:

- Complete information with filtration $\mathcal{F}^O$: Default times and X are observable.
- Incomplete information with filtration $\mathcal{F}^U$: Investors observe the time points of defaults and noisy observations of X, namely:

$$Z_t = \int_0^t a(X_s)ds + B_s,$$

where B is a standard d-dimensional Brownian motion, independent of all other processes and $a : S^X \to \mathbb{R}^d$.

Moreover, we set $\mathcal{F}^U = \mathcal{F}^{H,Z}$ and $\mathcal{F}^O = \mathcal{F}^{H,X,B}$ to obtain $\mathcal{F}^U \subset \mathcal{F}^O$.
We will use the superscripts U and O to distinguish the models.

Complete Information Model - Intensity based model

The complete information model is an example of an intensity based model. This class exhibits a lot of pricing formulas. For example

$$\mathbb{Q}(\tau_R > t_R, \tau_B > t_B, \tau_S > t_S \mid \mathcal{F}_t^O) = \prod_{i \in \{B,R,S\}} H_t^i \, \mathbb{E}\Big(\exp\Big(- \int_t^{t_i} \lambda_i(X_s) ds \Big) \mid X_t \Big) .$$

However, it is possible to find even closed form solution for many quantities:
Herefore, note that (X, H^B, H^R, H^S) is a finite state Markov chain and so many quantities can be expressed in terms of matrix exponentials of its generator.

Complete Information Model - Default probabilities

The default probabilities of R are given by

$$\mathbb{Q}\left(\tau_R \leq s \middle| X_t = k, H_t^R = 0\right) \quad = \quad 1 - e_k^\top e^{Q_R(s-t)} \mathbf{1}_K,$$

where

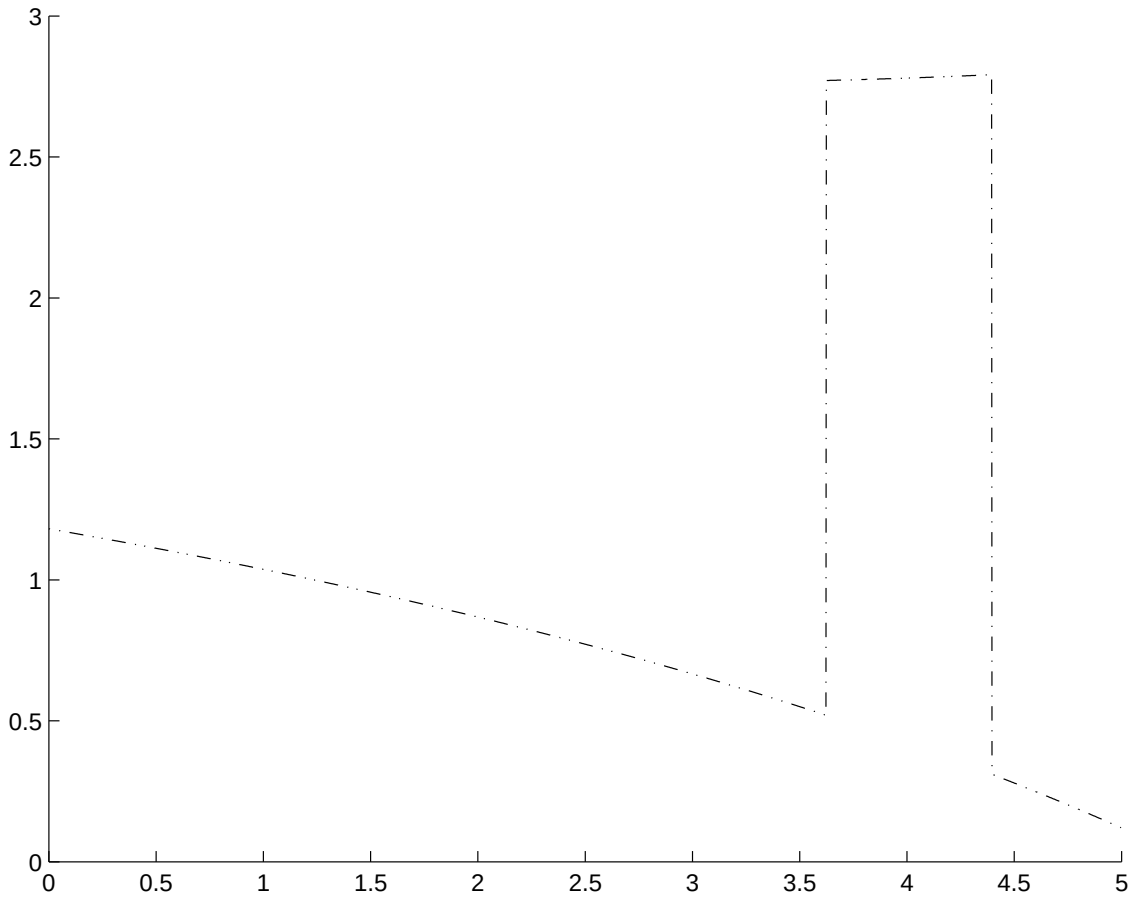
- $Q_R := W - \Lambda_R$ with $\Lambda_R = \mathrm{diag}(\lambda_R(1), \ldots, \lambda_R(K))$,
- $\mathbf{1}_K = (1, \ldots, 1)^\top$ is a column vector of dimension K and e_l denotes the l-th unit vector in $\mathbb{R}^K$.

By using the last result we obtain a pricing formula for risk-free CDS: P_t is a function of p^O of t and X_t. For $t \in [0, T]$ and $l \in \{1, \ldots, K\}$ we have

$$p^O(t, l) \quad := \quad \left(-\mathrm{LGD}_R\, e_l^\top Q_R - S e_l^\top\right) (Q_R - rI)^{-1} \left(e^{(Q_R - rI)(T-t)} - I\right) \mathbf{1}_K.$$

Trajectory

Trajectories of the fair CDS of R spread in the complete information model:



Complete Information Model - More formulas for default probabilities

- The distribution of the first-to-default time τ can be computed as:

$$\mathbb{Q}\left(\tau \leq s | X_t = k, H_t = 0\right) \quad = \quad 1 - e_k^\top e^{Q_{(1)}(s-t)} \mathbf{1}_K,$$

  where we defined $Q_{(1)} := W - \sum_{j \in \{B,R,S\}} \Lambda_j$.
- The probability that obligor $i \in \{B, R, S\}$ defaults first and that at τ the Markov chain is in the state l equals

$$\mathbb{Q}(X_\tau = l, \tau_i \leq s, \xi = i \mid X_t = k, H_t = 0)$$
$$= e_k^\top Q_{(1)}^{-1} \left(e^{Q_{(1)}(s-t)} - I \right) \Lambda_i e_l.$$

CCVA formula

Suppose that the collateral process satisfies $C_t = g(t, X_t)$ with a deterministic function g. Then, given $X_0 = j$,

$$\text{CCVA} = \sum_{k \in \{1,\dots,K\}} \int_0^T D(0,s) \Big(\text{LGD}_S \left(p^O(s,k)^+ - g(s,k)^+ \right)^+ + \text{LGD}'_S \left(g(s,k)^- - p^O(s,k)^- \right)^+ \Big) f^S_{j,k}(s) ds.$$

Note that the function $f^S_{j,k}$ is given by

$$\begin{aligned} f^S_{j,k}(s) &:= \frac{d}{ds} \mathbb{Q}(\tau \le s, \xi = S, X_\tau = k \mid X_0 = j) \\ &= e_j^\top e^{Q_{(1)}(s-t)} \Lambda_S e_k. \end{aligned}$$

CDVA formula

Similarly, given $X_0 = j$ we get for the CDVA:

$$\text{CDVA} = \sum_{k\in\{1,\dots,K\}} \int_0^T D(0,s) \Big(\text{LGD}_B \left(g(s,k)^- - p(s,k)^- \right)^- + \text{LGD}'_B \left(p(s,k)^+ - g(s,k)^+ \right)^- \Big) f^B_{j,k}(s) ds.$$

Note that the functions $f^B_{j,k}$ is given by

$$\begin{aligned} f^B_{j,k}(s) &:= \frac{d}{ds} \mathbb{Q}(\tau \le s, \xi = B, X_\tau = k \mid X_0 = j) \\ &= e_j^\top e^{Q_{(1)}(s-t)} \Lambda_B e_k. \end{aligned}$$

So we have a semi-closed formula for m.

Incomplete Information Model - Pricing

The following process will play an important role:

$$\pi_t^k := \mathbb{Q}(X_t = k \mid \mathcal{F}_t^U) \text{ for } 1 \leq k \leq K, \quad \text{and let } \pi_t := (\pi_t^1, \ldots, \pi_t^K)^\top.$$

It is possible to recover pricing formulas for contingent claims with H-adapted cashflows. For a contingent claim with H-adapted cashflows we introduce:

- $G(t,s)$ - cumulative discounted cashflows from t to s,
- p_t^U - incomplete information model price,
- $p^O(t, X_t)$ - complete information model price given X_t.

We obtain the following relationship:

$$p_t^U = \mathbb{E}\left(G(t,T)\middle|\mathcal{F}_t^U\right) = \mathbb{E}\left(\mathbb{E}\left(G(t,T)\middle|\mathcal{F}_t^O\right)\middle|\mathcal{F}_t^U\right) = \mathbb{E}\left(p^O(t,X_t)\middle|\mathcal{F}_t^U\right)$$
$$= \sum_{k=1}^{K} p^O(t,k)\pi_t^k.$$

Incomplete Information Model - Pricing of CDS

By using the last result we obtain a pricing formula for riskfree CDS:

$$P_t^U = \left(-\mathrm{LGD}_R\,\pi_t^\top Q_R - S\pi_t^\top\right)(Q_R - rI)^{-1}\left(e^{(Q_R-rI)(T-t)} - I\right)\mathbf{1}_K.$$

However, the approach does not work for non-H-adapted cashflows like the CCVA or CDVA. Therefore we have to rely on simulation techniques and one has to find the dynamics of π, which are derived in [Frey and Schmidt 2012].

Notation

We need the following notation: We denote the optional projection of a process $G = (G_t)_{t\in[0,T]}$ with respect to $\mathbb{F}^U$ by $\widehat{G}$. We have

$$\widehat{\lambda_{t,i}} := \widehat{\lambda_i(X_t)} = \mathbb{E}\left(\lambda_i(X_t)\middle|\mathcal{F}_t^U\right) = \sum_{j=1}^{K}\lambda_i(j)\pi_t^j.$$

$$\widehat{a(X_t)} = \mathbb{E}\left(a(X_t)\middle|\mathcal{F}_t^U\right) = \sum_{j=1}^{K}a(j)\pi_t^j$$

Moreover, we introduce the $\mathbb{F}^U$-Brownian motion μ defined by

$$\mu_t = (\mu_t^1,\ldots,\mu_t^d) \text{ with } \mu_t^i = Z_t^i - \int_0^t \widehat{a(X_s)}ds.$$

Kusher-Stratonovich-Equation

$$d\pi_t^k = \sum_{i=1}^{K} w_{ik}\pi_t^i dt + \sum_{j\in\{R,B,S\}} \left(\gamma_j^k(\pi_{t-})\right)^\top d\left(H_t^j + (1-H_t^j)\widehat{\lambda}_{t,j}dt\right) + \left(\alpha^k(\pi_t)\right)^\top d\mu_t\,,$$

$k = 1,\ldots,K$, where

$$\begin{aligned}\gamma_j^k(\pi_t) &= \pi_t^k\left(\frac{\lambda_j(k)}{\sum_{i=1}^K \lambda_j(i)\pi_t^i} - 1\right) \text{ for } 1\le j\le K \text{ and}\\ \alpha^k(\pi_t) &= \pi_t^k\left(a(k) - \sum_{i=1}^K \pi_t^i a(i)\right).\end{aligned}$$

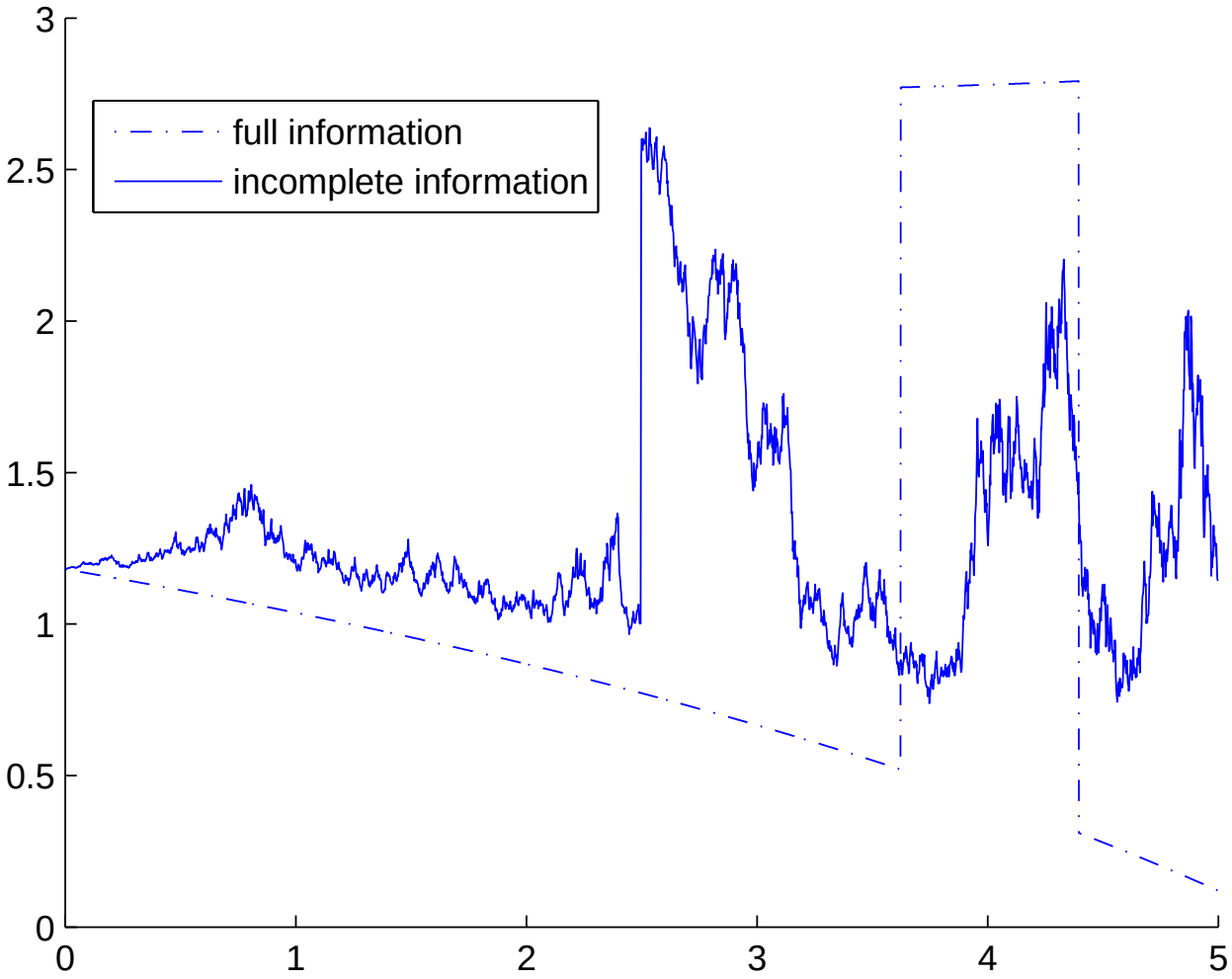
Contagion effects

$$\widehat{\lambda}_{\tau_j,i} - \widehat{\lambda}_{\tau_j-,i} = \sum_{k=1}^{K} \lambda_i(k)\pi^k_{\tau_j-} \left(\frac{\lambda_j(k)}{\sum_{l=1}^{K} \lambda_j(l)\pi^l_{\tau_j-}} - 1 \right) = \frac{\mathrm{cov}^{\pi_{\tau_j-}}(\lambda_i, \lambda_j)}{E^{\pi_{\tau_j-}}(\lambda_j)}.$$

An inspection of the upper formula shows that:

- Contagion effects are inversely proportional to the instantaneous default risk of the defaulting entity (firm j): a default of an entity with a better credit quality comes as a bigger surprise and the market impact is larger.
- Contagion effects are proportional to the covariance of the default intensities $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$ under the 'a-priori distribution' π_{τ_j-}. In particular, contagion effects are relatively high if the firms have similar characteristics in the sense that the functions $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$ are (almost) linearly dependent.

Trajectory of the fair CDS spread of R



Pricing of other contingent claims

Pricing of not H-adapted contingent claims like the CCVA and CDVA is more complicated and one has to rely on other algorithms like Monte Carlo simulation. The following algorithm is suggested in [Frey and Schmidt 2012] to simulate trajectories.

(1) Generate a trajectory of the Markov chain X.

(2) Generate for the trajectory of X constructed in (1) a trajectory of the default indicator H and the noisy information Z.

(3) Solve the system of SDEs numerically, for instance via a Euler-Maruyama type method.

Relationship between CCVA in the 2 models

By using Jensen's inequality one can show that for the collateralization strategy $C \equiv 0$ the following is true:

$$\text{CCVA}^O \geq \text{CCVA}^U \text{ and } \text{CDVA}^O \geq \text{CDVA}^U .$$

Simulation study

We calibrate the model to the following credit spreads in base points and default correlations in percentage points:

	B	R	S
Spread	50	1000	500

	ρ_{BR}	ρ_{BS}	ρ_{RS}
Correlation	2.0	1.5	5.0

Different credit valuation adjustments in bp for threshold collateralization for $\gamma = 0$ and $M_1 = M_2 =: M$ in the base scenario.

threshold	CCVA	CDVA
$M = 0$	0	0
$M = 0.02$	16	0
$M = 0.05$	38	1
no coll	93	1

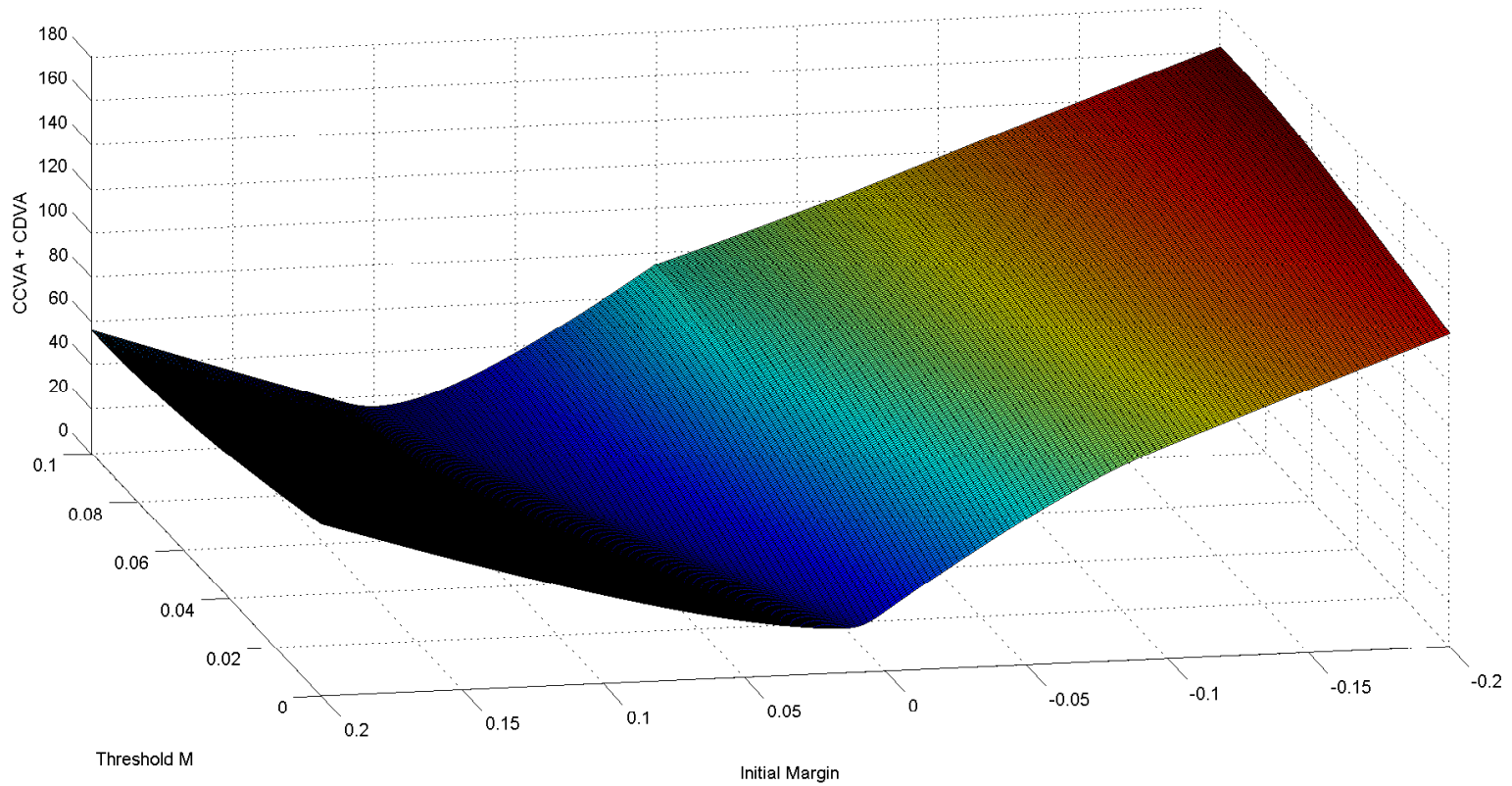
Table: Complete information model

threshold	CCVA	CDVA
$M = 0$	35	0
$M = 0.02$	45	0
$M = 0.05$	60	0
no coll	83	1

Table: Incomplete information model

Impact of γ

Graph of $m(C^{\gamma,M_1,M_2})$ with $M_1 = M_2 = M$ under incomplete information:



For example: $m(C^{0.01;0;0}) = 15bp$.

Improving Collateralization Strategies

We assume that B and S can agree on a strategy C which minizes $m(C) := CCVA(C) + CDVA(C)$. Recall the definitions:

$$\begin{aligned}
\mathrm{CCVA}(T,C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=S\}}D(0,\tau) \quad (\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ \\
&\qquad\qquad + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+)\big) \\
\mathrm{CDVA}(T,C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=B\}}D(0,\tau) \quad (\mathrm{LGD}_B(C_{\tau-}^- - P_\tau^-)^- \\
&\qquad\qquad + \mathrm{LGD}'_B(P_\tau^+ - C_{\tau-}^+)^-)\big).
\end{aligned}$$

We want to find an $\mathcal{F}$-adapted RCLL process C which minimizes $m(C)$.

Improving Collateralization Strategies - Complete Information model

Consider the collateralization strategy given by $C_t = P_t$:

- C coincides with the threshold strategy given by $\gamma = 0$ and $M_1 = M_2 = 0$,
- $C_{\tau-} = P_{\tau-} = P_\tau$ Q-a.s. and therefore $\mathrm{CCVA}(C) = 0$ and $\mathrm{CDVA}(C) = 0$.

Therefore C is a collateralization strategy which minizes m.

Improving Collateralization Strategies - Incomplete Information model

- Contagion effects $\Rightarrow P_{\tau-} < P_{\tau}$ Q-a.s.
- Threshold collateralization with $\gamma = 0$ systematically underestimates the price of the CDS.

Is it possible to find an improvement?

Improving Collateralization Strategies - Incomplete Information model

In [Frey and Rösler 2014] a minimizer of $C \mapsto m(C)$ in the incomplete information model is found. It depends only on the quantities:

$$d_j(\pi_{\tau-}) := \mathbb{Q}\left(\xi = j | \mathcal{F}_{\tau-}\right) = \frac{(\widehat{\lambda}_j)_{\tau-}}{\sum_{i\in\{B,R,S\}}(\widehat{\lambda}_i)_{\tau-}},$$

$$x_j(\tau, \pi_{\tau-}) := p^U\left(\tau, \pi_{\tau-} + \operatorname{diag}\left(\gamma_j^1, \ldots, \gamma_j^K\right)\pi_{\tau-}\right)$$

for $j \in \{B, S\}$.

CCVA	CDVA	CCVA + CDVA
0	0	0

Table: Effectivity of improved strategy

Conclusion

Our findings:

- Threshold-collateralization with initial margin $\gamma = 0$ works fine in the complete information model.
- The presence of contagion effects lead to an underestimation of the price of risk-free CDS in threshold collateralization strategies.
- The use of the refined strategy, altough possibly model-dependent, may be reasonable.

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