Inflation-indexed Swaps and Swaptions

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Outline

- Introduction: Markets, Instruments & Literature
- Foreign-Exchange Analogy
- Pricing & Hedging of Inflation Swaps
- Pricing Inflation Swaptions with a Market Model

Overview

Inflation

- An increase in the economy's price level is known as inflation. Inflation reduces the purchasing power, i.e. the value of money decrease.
- A consumer price index (CPI) is the price of a particular basket consisting of consumer goods and services. The price index is a measure of the general price level in the economy.
- Inflation is typically measured as the percentage rate at which the consumer price index changes over a certain period of time.
- Negative inflation is known as deflation.

Inflation Protected Bonds

- Many names: Inflation-indexed bonds, Inflation linked bonds, Real bonds, TIPS (US), Index-linked gilts (UK).
- The payoff is linked to a price index. (CPI, RPI)
- Typically coupon bonds.
- Can be floored.
- The issuer of an inflation protected bond has an incentive to keep inflation low. Useful for governments.
- These bonds are typically issued by Treasuries.
- Typical investors are pension funds, mutual funds.
- World wide outstanding nominal amount 2007: 1000 billion dollar.

Markets

- UK (1981)
- Australia (1983)
- Canada (1991)
- Sweden (1994)
- United States (1997)
- Greece (1997)
- France (1998)
- Italy (2003)
- Japan (1904)
- Germany (2006)

Earlier: Chile, Brazil, Columbia, Argentina.

First inflation protected bond issue: Massachusetts Bay Company 1780.

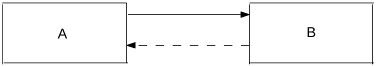
Inflation Derivatives

- Swaps
- Caps & Floors
- Swaptions
- Bond options
- ...

Inflation Indexed Swaps & Swaptions

Inflation Indexed Swap

- Agreement between two parties A and B to exchange cash flows in the future
- Prespecified dates for when the cash flows are to be exchanged
- At least one of the cash flows is linked to inflation (CPI)



Inflation Indexed Swaption

- It is an *option* to enter into an *inflation indexed swap* at pre specified date at a pre determined swap rate.

Main References

Hughston (1998)

- General theory
- Foreign-currency analogy

Jarrow & Yildirim (2003)

- 3-factor HJM model
- TIPS (coupon bonds)
- Option on Inflation index

Mercurio (2005)

- YYIIS, Caplets, Floorlets (ZCIIS)
- JY version of HJM with Hull-White vol
- 2 Market Models

Contribution

- HJM model with jumps
  - YYIIS
- Inflation Swap Market Models
  - ZCIISwaptions
  - YYIISwaptions
- HJM model
  - ZCIISwaptions
  - TIPStions
- Verify the foreign-currency analogy for an arbitrary process

YYIIS= Year-on-Year Inflation Indexed Swaps

ZCIIS= Zero Coupon Inflation Indexed Swaps

Overview

Price and Payoff

$\mathbf{I(t)}$: An arbitrary stochastic process

$\mathbf{p^n(t,T)}$: Price in dollar at t of a contract that pays out 1 dollar at T.

$\mathbf{p^{IP}(t,T)}$: Price in dollar at t of a contract that pays out $I(T)$ dollar at T.

Assume : There exist a market for $p^n(t,T)$ and $p^{IP}(t,T)$ for all T

Define : $p^r(t,T) = \frac{p^{IP}(t,T)}{I(t)}$

If $I(t)$ is the price of a hamburger

A nominal bond:

- Pays out 1 dollar at maturity.
- $p^n(t,T)$: the price of a nominal bond is in dollar

A hamburger-indexed bond:

- At maturity it pays out a dollar amount that is enough to buy 1 hamburger.
- $p^{IP}(t,T)$: the price of a hamburger-inflation protected bond is in dollar

A hamburger-real bond:

- Pays out 1 hamburger at maturity
- $p^r(t,T)$: the price of a real bond is in hamburgers

Note: CPI

Define

Forward rates: $f^i(t,T) = -\frac{\partial \ln p^i(t,T)}{\partial T} \quad \text{for } i = r, n.$

Short rates: $r^i(t) = f^i(t,t) \quad \text{for } i = r, n.$

Money Market Accounts: $B^i(t) = e^{\int_0^t r^i(s)ds} \quad \text{for } i = r, n.$

HJM model with Jumps

Assume:
Under the objective probability measure P:

$$
\begin{aligned}
df_t^r(T) &= \alpha_t^r(T)dt + \sigma_t^r(T)dW^P + \int_V \xi^r(t,v,T)\mu^P(dt,dv) \\
df_t^n(T) &= \alpha_t^n(T)dt + \sigma_t^n(T)dW^P + \int_V \xi^n(t,v,T)\mu^P(dt,dv) \\
dI_t &= I_t\mu_t^I dt + I_t\sigma_t^I dW^P + I_{t-}\int_V \gamma_t^I(v)\mu^P(dt,dv)
\end{aligned}
$$

Now calculate

1. Forward rates $\Rightarrow$ Bondprices (BKR)
2. Change measure from P to Q^n (Girsanov)
   Now we have found the Q^n-drift of $p^n(t,T)$ and $p^{IP}(t,T)$ which we call $\mu_Q^n(t,T)$ and $\mu_Q^{IP}(t,T)$
3. By requiring

$$\frac{p^n(t,T)}{B^n(t)} \quad \frac{p^{IP}(t,T)}{B^n(t)} \quad \text{are } Q^n\text{-martingales}$$

i.e. $\mu_Q^n(t,T) = \mu_Q^{IP}(t,T) = r^n(t)$ for all maturities T.
$\Downarrow$
3 drift conditions
$\Downarrow$
One of the 3 conditions tells us that the Q^n-drift of the index I is equal to $r^n - r^r$

Three drift conditions

$$
\begin{aligned}
\alpha^n(t,T) &= \sigma^n(t,T)\left(\int_t^T \sigma^r(t,s)ds - h(t)\right) \\
&- \int_V \{\delta^n(t,v,T)+1\}\,\xi^n(t,v,T)\lambda_t(dv)
\end{aligned}
$$

$$
\begin{aligned}
\alpha^r(t,T) &= \sigma^r(t,T)\left(\int_t^T \sigma^r(t,s)ds - \sigma^I(t) - h(t)\right) \\
&- \int_V \left(1+\gamma^I(t,v)\right)\left(1+\delta^r(t,v,T)\right)\xi^r(t,v,T)\lambda_t(dv)
\end{aligned}
$$

$$
\mu^I(t) = r^n(t) - r^r(t) - h(t)\sigma^I(t) - \int_V \gamma^I(t,v)\lambda_t(dv)
$$

Result

Under the nominal risk neutral measure Q^n:

$$\begin{aligned}
\frac{dp_t^n(T)}{p_{t-}^n(T)} &= r_t^n dt + \beta_t^n(T)dW + \int_V \delta_t^n(v,T)\tilde{\mu}(dt,dv) \\
\frac{dp_t^{IP}(T)}{p_{t-}^{IP}(T)} &= r_t^n dt + \beta_t^{IP}(T)dW + \int_V \delta_t^{IP}(v,T)\tilde{\mu}(dt,dv) \\
\frac{dI_t}{I_{t-}} &= (r_t^n - r_t^r)dt + \sigma_t^I dW + \int_V \gamma_t^I(v)\tilde{\mu}(dt,dv) \\
\frac{dp_t^r(T)}{p_{t-}^r(T)} &= a(t,T)dt + \beta_t^r(T)dW + \int_V \delta_t^r(v,T)\tilde{\mu}(dt,dv)
\end{aligned}$$

where

$$\tilde{\mu}(dt,dv) = \mu(dt,dv) - \lambda_t(dv)dt$$

Note: I has the same dynamics as an FX-rate!

Foreign Currency Analogy

Nominal vs Real

$p^n(t,T)$:	Price of nominal T-bond in dollar
$p^r(t,T)$:	Price of real T-bond in CPI units*
$I(t)$:	Price level (dollar per CPI-unit)
$p^{IP}(t,T)$:	Price of a real T-bond in dollar denoted by $p^{IP}(t,T)$

Domestic vs Foreign

$p^n(t,T)$:	Price of domestic T-bond
$p^r(t,T)$:	Price of foreign T-bond
$I(t)$:	FX-rate (domestic per foreign unit)
$I(t)p^r(t,T)$:	Domestic price of foreign T-bond *

Overview

1. Introduction to the Inflation Market & Instruments
2. Foreign-Exchange Analogy
3. **Pricing & Hedging of Inflation Swaps**
4. Pricing Inflation Swaptions with a Market Model

Assumptions

- Exist a market for nominal T-bonds and nominal indexed-bonds for all maturity dates.
- The bond prices are differentiable wrt T.
- Forward rate dynamics according to HJM with jumps.
- Existence of martingale measure.
- All volatilities and the intensity are deterministic under the nominal risk neutral measure.

A Payer Swap

- starts at time T_m
- At each payment date T_j where $j = m+1,\ m+2,\ \cdots\ ,\ T_M$
  - you pay
$$\alpha_j K$$
  - you receive
$$\alpha_j \left[\frac{I(T_j)}{I(T_{j-1})} - 1 \right]$$

The price

The price at time t is:

$$\sum_{j=m+1}^{M} \Pi\left[t, \alpha_j \frac{I(T_j)}{I(T_{j-1})}\right] - (K+1) \sum_{j=m+1}^{M} \alpha_j p(t, T_j)$$

Find:

$$\Pi\left[t, \alpha_j \frac{I(T_j)}{I(T_{j-1})}\right]$$

i.e. the price of payoff

$$\alpha_j \frac{I(T_j)}{I(T_{j-1})}$$

Tool

Let $M(t)$ and $N(t)$ be two martingales so that

$$E_t[M_T] = M_t$$
$$E_t[N_T] = N_t$$

The key to price this swap is to find:

$$E_t[M_T N_T]$$

Solution:

- If $M(t)$ and $N(t)$ are independent, then $E_t[M_T N_T] = M_t N_t$
- What if they are NOT independent?
  $E_t[M_T N_T] = M_t N_t G_t^T$ where G_t^T is the "convexity correction".

Basic Result

Let $M(t)$ and $N(t)$ be two martingales with dynamics:

$$\frac{dM_t}{M_{t-}} = \sigma_t^M dW_t + \int_V \delta_t^M(v)\tilde{\mu}(dt, dv)$$

$$\frac{dN_t}{N_{t-}} = \sigma_t^N dW_t + \int_V \delta_t^N(v)\tilde{\mu}(dt, dv)$$

Assume that σ^M, σ^N, δ^M, δ^N, λ are deterministic.

Then

$$E_t[M_T N_T] = M_t N_t G_t^T$$

where

$$G_t^T = e^{\int_t^T \left(\sigma_u^M \cdot \sigma_u^N + \int_V \delta_u^M(v)\delta_u^N(v)\lambda_u(dv)\right)du}$$

The Inflation-linked Swap Leg

The payoff function

$$\mathcal{X}_2 = \frac{I(T_2)}{I(T_1)}$$

The value at time t

$$\Pi\left[t, \mathcal{X}_2\right] = p^n(t, T_1) E_t^{T_1,n}\left[p^r(T_1, T_2)\right]$$

where

$$E_t^{T_1,n}\left[p^r(T_1, T_2)\right] = E_t^{T_1,r}\left[\frac{p^r(T_1, T_2)}{p^r(T_1, T_1)}\frac{L(T_1)}{L(t)}\right] = \frac{p^r(t, T_2)}{p^r(t, T_1)} C(t, T_1, T_2)$$

hence

$$\Pi\left[t, \mathcal{X}_2\right] = \frac{p^n(t, T_1) p^r(t, T_2)}{p^r(t, T_1)} C(t, T_1, T_2)$$

where $C(t, T_1, T_2)$ is a convexity correction term.

The Inflation-linked Swap Leg

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The value at time t

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The Inflation-linked Swap Leg

The payoff function

$$\mathcal{X}_2 = \frac{I(T_2)}{I(T_1)}$$

The value at time t

$$\Pi\left[t, \mathcal{X}_2\right] = p^n(t, T_1) E_t^{T_1,n}\left[p^r(T_1, T_2)\right]$$

where

$$E_t^{T_1,n}\left[p^r(T_1, T_2)\right] = E_t^{T_1,r}\left[\frac{p^r(T_1, T_2)}{p^r(T_1, T_1)} \frac{L(T_1)}{L(t)}\right] = \frac{p^r(t, T_2)}{p^r(t, T_1)} C(t, T_1, T_2)$$

hence

$$\Pi\left[t, \mathcal{X}_2\right] = \frac{p^n(t, T_1) p^r(t, T_2)}{p^r(t, T_1)} C(t, T_1, T_2)$$

where $C(t, T_1, T_2)$ is a convexity correction term.

The Payer Swap

The price of the swap is at time t

$$\sum_{j=m+1}^{M} \alpha_j \frac{p^n(t,T_{j-1})p^{IP}(t,T_j)C(t,T_{j-1},T_j)}{p^{IP}(t,T_{j-1})}$$

$$- \quad (K+1) \sum_{j=m+1}^{M} \alpha_j p^n(t,T_j)$$

Note
The price does only depend on bonds in the nominal market.

Hedging the Inflation-linked Swap leg I

When no jumps

$$\Pi\left[t, \mathcal{X}_2\right] = \frac{p^n(t,T_1)p^{IP}(t,T_2)}{p^{IP}(t,T_1)} e^{g(s,T_1,T_2)}$$

where

$$g(s,T_1,T_2) = \int_t^{T_1} \left(\beta^n(s,T_1) - \beta^{IP}(s,T_1)\right) \cdot \left(\beta^{IP}(s,T_2) - \beta^{IP}(s,T_1)\right) ds$$

Idea:
Try to replicate the swap leg using $p^n(t,T_1), p^{IP}(t,T_2)$ and $p^{IP}(t,T_1)$ from now on referred to as S_1, S_2 and S_3 respectively.

Hedging the Inflation-linked Swap leg II

Define portfolio strategy $(h_1(t), h_2(t), h_3(t))$ for $t \leq T_1$ as

$$h_i(t) = \frac{\Pi\left[t, \mathcal{X}_2\right]}{S_i(t)} \quad \text{for } i = 1, 2, \quad h_3(t) = -\frac{\Pi\left[t, \mathcal{X}_2\right]}{S_3(t)}$$

Then for $t \leq T_1$

$$\begin{aligned} V^h(t) &= \sum_1^3 h_i S_i = \Pi(t) \\ dV^h(t) &= \sum_1^3 h_i dS_i \end{aligned}$$

At time T_1 we have $V^h(T_1) = \frac{p^{IP}(T_1,T_2)}{I(T_1)}$ which is just enough to buy $\frac{1}{I(T_1)}$ $T_2 - IP$-bonds which we keep until maturity and thus results in $\frac{I(T_2)}{I(T_1)}$ as it should!

Overview

The Swap rate

Recall that the swap price is:

$$YYIIS_m^M(t,K) = \sum_{j=m+1}^{M} \Pi\left[t, \alpha_j \frac{I(T_j)}{I(T_{j-1})}\right] - (K+1)S_m^M(t))$$

where

$$S_m^k(t) = \sum_{j=m+1}^{k} \alpha_j p_n(t, T_j)$$

The par swap rate is:

$$R_m^M(t) = \frac{\sum_{j=m+1}^{M} \Pi\left[t, \alpha_j \frac{I(T_j)}{I(T_{j-1})}\right] - S_m^M(t)}{S_m^M(t)}$$

The HJM Swap rate

The par swap rate at time t

$$R_m^M(t) = \frac{\sum_{j=m+1}^{M} \frac{\alpha_j p^n(t,T_{j-1}) p^{IP}(t,T_j) C(t,T_{j-1},T_j)}{p^{IP}(t,T_{j-1})} - S_m^M(t)}{S_m^M(t)}$$

Where

$$S_m^k(t) = \sum_{j=m+1}^{k} \alpha_j p_n(t, T_j)$$

Note

Nasty distribution!

A payer YYIISwaption

Payoff function

$$\Upsilon_m^M = max[YYIIS_m^M(T_m, K), 0]$$

Rewritten Payoff function

$$\Upsilon = S_T max[R_T - K, 0]$$

where
R_T par swap rate (self-financing portfolio)
S_T sum of nominal bonds (self-financing portfolio)

Note
Easy if R_T is lognormal! $\Rightarrow$ Black's pricing formula

Swap Market Model

Definition:
For any given pair (m,k) of integers s.t. $0 \leq m < k < M$ we assume that, under the measure for which S_m^k is numeraire, the forward swap rate R_m^k has dynamics given by

$$dR_m^k(t) = R_m^k(t)\sigma_m^k(t)dW_m^k(t)$$

where
$\sigma_m^k(t)$ is deterministic

Summarizing

- Introduction: Markets, Instruments & Literature
- Foreign-Exchange Analogy
- Pricing & Hedging of Inflation Swaps
- Pricing Inflation Swaptions with a Market Model

Remarks

- TIPS can have embedded option features
- Reverse: Swaps as given price TIPS
- The CPI index used is typically lagged
- The CPI index is typically only observed monthly (linearly interpolated)
- The suggested market model is not proved to exist