On Optimal Dividends and Capital Injection Problems in Risk Theory

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The Basic Models

The Classical Risk Model

\[ X_t^0 = x + ct - \sum_{i=1}^{N_t} Y_i \]

- \( x \): initial capital
- \( c \): premium rate
- \( \{ N_t \} \): Poisson process with rate \( \lambda \)
- \( \{ Y_i \} \): iid, independent of \( \{ N_t \} \)
- \( G(y) \): continuous distribution function of \( Y_i \), \( G(0) = 0 \)
- \( \mu_n = \mathbb{E}[Y_i^n], \quad \mu = \mu_1 \)
Diffusion Approximations

Consider the sequence of classical risk processes

\[ X_t^{(n)} = x + \left( 1 + \frac{c/(\lambda \mu) - 1}{\sqrt{n}} \right) (\lambda n) \frac{\mu}{\sqrt{n}} t - \sum_{i=1}^{N_{nt}} \frac{Y_i}{\sqrt{n}}. \]

They all have the same first two moments as the original process. \( \{X_t^{(n)}\} \) converges weakly to

\[ X_t = x + (c - \lambda \mu)t + \sqrt{\lambda \mu_2} W_t, \]

where \( \{W_t\} \) is a standard Brownian motion. We call \( \{X_t\} \) a diffusion approximation to \( \{X_t^0\} \).
Let \( \{D_t\} \) be an increasing cadlag process with \( D_0^- = 0 \). For a stochastic process \( \{X^0_t\} \) let \( X^D_t = X^0_t - D_t \) be the post dividend process. By \( \tau^D = \inf\{t : X^D_t < 0\} \) we denote the time of ruin. The value of the dividend strategy \( \{D_t\} \) is

\[
V^D(x) = \mathbb{E}\left[ \int_{0^-}^{\tau^D_-} e^{-\delta t} \, dD_t \right],
\]

where \( \delta > 0 \).

We are interested in \( V(x) = \sup_D V^D(x) \) and — if it exists — the optimal strategy.
Gerber’s Solution in the Classical Case

For a classical risk model Gerber has shown — via discretisation — that the optimal strategy is a band strategy.
### Introduction

- **De Finetti’s Dividend Problem**
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- **Dividends and Capital Injections**
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- **Measuring Risk via Capital Injections**
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**On Optimal Dividends and Capital Injection Problems in Risk Theory**
The Diffusion Approximation

Restricted Dividend Payments

We consider a process $X^0_t = x + mt + \sigma W_t$.
Suppose only dividends of the form $D_t = \int_0^t d_s \, ds$ with $0 \leq d_s \leq u$ are allowed. Then one expects the value function $V(x)$ to fulfil

$$\sup_{0 \leq d \leq u} \left\{ \frac{1}{2} \sigma^2 V''(x) + (m - d) V'(x) - \delta V(x) + d \right\} = 0 .$$

We know that $V(0) = 0$.
We choose $d(x) = 0$ if $V'(x) \geq 1$ and $d(x) = u$ if $V'(x) < 1$.
Thus

$$\max\{\frac{1}{2} \sigma^2 V''(x) + m V'(x) - \delta V(x), \frac{1}{2} \sigma^2 V''(x) + (m - u) V'(x) - \delta V(x) + u\} = 0 .$$
We write

\[
\max \left\{ \frac{1}{2} \sigma^2 V''(x) + m V'(x) - \delta V(x), \right.
\]

\[
\left. \frac{\sigma^2}{2u} V''(x) + \left( \frac{m}{u} - 1 \right) V'(x) - \frac{\delta}{u} V(x) + 1 \right\} = 0.
\]

Letting \( u \to \infty \) yields

\[
\max \left\{ \frac{1}{2} \sigma^2 V''(x) + m V'(x) - \delta V(x), 1 - V'(x) \right\} = 0.
\]

We expect no dividend if \( V'(x) > 1 \), dividends, if \( V'(x) = 1 \).
For $x$ close to 0 we expect a barrier strategy. At the barrier $b$ we expect $V'(b) = 1$.

The equation $\frac{1}{2}\sigma^2 f''(x) + mf'(x) - \delta f(x) = 0$ with initial value 0 has the solution

$$f(x) = e^{\theta_1 x} - e^{\theta_2 x},$$

where $\theta_{1/2}$ solves

$$\frac{1}{2}\sigma^2 \theta^2 + m\theta - \delta = 0.$$

Barrier at $b$ determines $V(x)$ through $f'(b) = 1$; i.e.,

$$V(x) = f(x)/f'(b).$$

$V(x)$ should be maximal; that is, $b$ is determined through

$$f''(b) = 0.$$ In particular, $V(b) = m/\delta$. 
The Verification Theorem

We let $V(x) = m/\delta + (x - b)$ for $x \geq b$. Itô’s lemma shows that

$$\left\{ e^{-\delta(t^{\wedge}t)} V(X_{t^{\wedge}t}) - \int_0^{t^{\wedge}t} e^{-\delta s} \left[ \frac{1}{2} \sigma^2 V''(X_s) + mV'(X_s) - \delta V(X_s) \right] \, ds + \int_0^{t^{\wedge}t} e^{-\delta s} V'(X_s) \, dD_t \right\}$$

is a martingale. supermartingale.
The Verification Theorem

We conclude,

\[ V(x) \geq \mathbb{E}\left[ e^{-\delta(\tau \wedge t)} V(X_{\tau \wedge t}) + \int_0^{\tau \wedge t} e^{-\delta s} \, dD_t \right]. \]

Letting \( t \to \infty \), \( V(x) \geq V^D(x) \).

Let \( D^*_t = \max\{\sup_{0 \leq s \leq t} X_s^0 - b, 0\} \) and \( X^*_t = X_t^0 - D^*_t \). Then

\[ \left\{ e^{-\delta(\tau \wedge t)} V(X^*_{\tau \wedge t}) + \int_0^{\tau \wedge t} e^{-\delta s} \, dD^*_t \right\} \]

is a martingale. Thus \( V(x) = V^*(x) \).
Properties of $V(x)$

We have

$$\int_0^\infty e^{-\delta t} \, dD_t = \delta \int_0^\infty D_t e^{-\delta t} \, dt .$$

Let $\tilde{D}_t = x + ct$. Then $X\tilde{D} \leq 0$. Thus $D_t \leq \tilde{D}_t$ for all possible $D$. We get

$$V(x) \geq IE \left[ \int_0^{\tilde{D}} e^{-\delta t} \, d\tilde{D}_t \right] = x + \frac{c}{\lambda + \delta} .$$

$$V(x) \leq \int_0^\infty e^{-\delta t} \, d\tilde{D}_t \leq x + \frac{c}{\delta} .$$

$V(x)$ is locally Lipschitz-continuous and therefore absolutely continuous.
The Hamilton–Jacobi–Bellman Equation

Very technical considerations yield in the classical model the equation

$$\max \left\{ cV'(x) + \lambda \left[ \int_0^x V(x - y) \, dG(y) - V(x) \right] - \delta V(x), \quad 1 - V'(x) \right\} = 0.$$ 

$V'(x)$ has only upward jumps.
The Classical Model

The Optimal Strategy

- If $V'(x-)>1$ do not pay dividends.
- On the upper boundary of regions where no dividend is paid, pay the incoming premium as dividend.
- At other points pay the minimal amount to reach a point, where the incoming premium is paid as dividend.
The function \( f(x) = x + \kappa \) with \( \kappa \geq c/\delta \) solves the Hamilton–Jacobi–Bellman equation.

\( V(x) \) is — even for nice claim size distributions — not necessarily differentiable.
The Classical Model

Characterisation of the Value Function

**Theorem**

$V(x)$ is the minimal solution to the Hamilton–Jacobi–Bellman equation.

If $f(x)$ is a solution such that $f(x)e^{-\delta x/c} \to 0$ as $x \to \infty$, $f'(x)$ has only upward jumps and either $f'(0) > 1$ or $f(0) = c/(\lambda + \delta)$ then $f(x) = V(x)$. 
The Problem

In contrast to de Finetti’s dividend problem we do not stop at ruin but the share holders have cover the deficit. If we denote the cumulative injections up to time \( t \) by \( Z_t \), the surplus is \( X^{D,Z}_t = X^0_t - D_t + Z_t \). A pair \( (D, Z) \) is admissible, if \( X^{D,Z}_t \geq 0 \) for all \( t \). The value of a strategy is

\[
V^{D,Z}(x) = \mathbb{E} \left[ \int_0^\infty e^{-\delta t} \, dD_t - \phi \int_0^\infty e^{-\delta t} \, dZ_t \right],
\]

where \( \phi > 1 \). \( V(x) = \sup_{D,Z} V^{D,Z}(x) \).

\( \phi < 1 \) yields an infinite value function.

\( \phi = 1 \) has the trivial solution, where \( X^{*}_t \equiv 0 \).
It cannot be optimal to inject capital before it really is necessary. Thus, for a strategy $D$ we choose

$$Z_t = \max \left\{ \sup_{0 \leq s \leq t} \{D_s - X_s^0\}, 0 \right\}.$$  

We only have to choose $D$. 
Concavity of the Value Function

Consider strategies \( \{D_t^x\} \) and \( \{D_t^y\} \) for initial capitals \( x \) and \( y \), respectively. Let \( \alpha \in (0, 1) \) and \( \beta = 1 - \alpha \). Then

\[
\alpha x + \beta y + X_t^0 - (\alpha D_t^x + \beta D_t^y) + (\alpha Z_t^x + \beta Z_t^y) = \alpha X_t^x + \beta X_t^y \geq 0.
\]

Thus \( Z_t^{\alpha x+\beta y} \leq \alpha Z_t^x + \beta Z_t^y \), and

\[
V(\alpha x + \beta y) \geq V^{\alpha x+\beta y}(\alpha x + \beta y) \geq \alpha V^x(x) + \beta V^y(y).
\]

Maximizing the right-hand side yields

\[
V(\alpha x + \beta y) \geq \alpha V(x) + \beta V(y).
\]
Properties of the Value Function

Let \( x > y \geq 0 \). Then \( V(x) \geq V(y) + (x - y) \). Thus \( V'(x) \geq 1 \).

\[ V(y) \geq V(x) - \phi(x - y) \]. Thus \( V'(x) \leq \phi \).

\( V'(x) \) is decreasing with \( V'(0+) \leq \phi \). If it exists let \( b = \inf \{ x : V'(x) = 1 \} \). Then \( V(x) = V(b) + x - b \) for \( x \geq b \).
We expect $V(x)$ to solve

$$\max\left\{ \frac{1}{2}\sigma^2 V''(x) + m V'(x) - \delta V(x), 1 - V'(x) \right\} = 0.$$ 

The solution of the first part is of the form

$$V(x) = Ae^{\theta_1 x} - Be^{\theta_2 x}.$$ 

The constants $A, B, b$ are determined through $V''(b) = 0,$ $V'(b) = 1$ and $V'(0) = \phi.$

A verification theorem can be proved.
The Hamilton–Jacobi–Bellman Equation

Technical considerations give that $V(x)$ solves

$$
\max \left\{ cV'(x) + \lambda \left[ \int_0^x V(x - y) \, dG(x) \right] + \int_x^\infty (\phi(x - y) + V(0)) \, dG(y) \right\} - (\lambda + \delta)V(x),

1 - V'(x) \right\} = 0.
$$

For the initial condition we find

$$
V(0) = \frac{cV'(0) - \phi \lambda \mu}{\delta} \leq \frac{(c - \lambda \mu) \phi}{\delta}.
$$
We cannot expect that there is a unique solution.

**Theorem**

\[ V(x) \text{ is the minimal solution to the Hamilton–Jacobi–Bellman equation.} \]

If \( f(x) \) is a concave solution and either \( f'(0) > 1 \) or \( f(x) = x + (c - \phi \lambda \mu) / \delta \) then \( f(x) = V(x) \).
If it is optimal to pay all capital as dividend then

\[ V(x) = x + \frac{c - \phi \lambda \mu}{\delta}, \quad x \geq 0. \]

The Hamilton–Jacobi–Bellman equation is fulfilled if and only if

\[ \int_0^x \left[ \lambda (\phi - 1)(1 - G(y)) - \delta \right] dy \leq 0. \]

Thus this is the solution if and only if \( \delta \geq \lambda (\phi - 1) \).

Note that the claim size distribution does not play any rôle.
Solution for Exponentially Distributed Claim Sizes

Suppose that $G(y) = 1 - e^{-\alpha y}$ and $\delta < \lambda(\phi - 1)$. The solution to

$$cV'(x) + \lambda \int_0^x V(x - y)\alpha e^{-\alpha y} \, dy + \left( V(0) - \frac{\lambda \phi}{\alpha} \right) e^{-\alpha x} - (\lambda + \delta)V(x) = 0$$

is of the form

$$V(x) = C_1 \cdot e^{v_1 x} + C_2 \cdot e^{v_2 x},$$

where $v_1$ and $v_2$ are the solutions to

$$cv^2 - (\lambda + \delta - \alpha c)v - \alpha \delta = 0.$$  

The equation and $V'(b) = 1$ give $C_k$ as functions of $b$. Maximising $V(0)$ yields $b$. 

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The Classical Model

Solution for $c = 4$, $\alpha = 3$, $\delta = 0.06$ and $\lambda = 2$
Further Problems

- The business stops the first time the injection process increases; i.e., only one injection is done.
- The investors can stop the business without capital injection; i.e., if the value is negative.
- At the time of an injection a fixed penalty has to be paid.
- The company can buy reinsurance and/or invest into a risky asset.
A classical measure for the risk are the ruin probabilities.

De Finetti proposed to measure the value of the future dividend payments.

Alternatively, one could measure the risk as the *value of the future capital injections*. We do not anymore allow dividend payments. That is, we consider

\[ V(x) = \mathbb{E} \left[ \int_{0}^{\infty} e^{-\delta t} \, dZ_t \right], \]

where \( \delta \geq 0 \).
The Optimisation Problem

It is again optimal to wait with the injections as long as possible.

In addition the insurer can now buy reinsurance in order to minimise the value. That is, for a premium rate $c - c(b)$ the part of the claim $Y_i$ covered by the insurer is $R(Y_i, b)$. The insurer can choose a strategy $\{b_t\}$, where $b_t \in [0, a]$ for $a \in (0, \infty]$. Here 0 means full reinsurance, $a$ means no reinsurance. We assume $c(0) < 0$. 
The Hamilton–Jacobi–Bellman Equation

The diffusion approximation to a controlled risk process is

\[ X_t^b = x + \int_0^t \{ c(b_s) - \lambda \mathbb{E}[R(Y, b_s)] \} \, ds \]

\[ + \sqrt{\lambda \mathbb{E}[R^2(Y, b_t)]} \, dW_s + Z_t . \]

We expect the minimal value \( V(x) \) to solve

\[ \inf_{b \in [0, a]} \left\{ \frac{1}{2} \lambda \mathbb{E}[R^2(Y, b)] V''(x) \right. \]

\[ + \left. \{ c(b) - \lambda \mathbb{E}[R(Y, b)] \} V'(x) - \delta V(x) \right\} = 0 . \]
Suppose we know the optimal strategy. Let $\tau = \inf\{t : X_t = 0\}$ be the time where $Z_t$ increases for the first time under the optimal strategy. Thus

$$V(x) = V(0)\mathbb{E}[e^{-\delta \tau}; \tau < \infty].$$

Instead of minimising $V^b(x)$ we can minimise $\mathbb{E}[e^{-\delta \tau}; \tau < \infty]$. 
The Form of the Value Function

Starting at $x + y$ we first have to reach $x$ before reaching $0$. Under the optimal strategy we find

$$\mathbb{E}[\exp\{-\delta \tau^x+y\}] = \mathbb{E}[\exp\{-\delta (\tau^y + \tilde{\tau}^x)\}]$$

$$= \mathbb{E}[\exp\{-\delta \tau^y\}] \mathbb{E}[\exp\{-\delta \tau^x\}].$$

Thus we conclude that $V(x)$ is an exponential function and the optimal strategy is constant.
The Value Function

Using \( V(x) = Ce^{-rx} \), we find from the Hamilton–Jacobi–Bellman equation

\[
\inf_{b \in [0,a]} \left\{ \frac{1}{2} \lambda \mathbb{E}[R^2(Y, b)]r^2 - \{c(b) - \lambda \mathbb{E}[R(Y, b)]\}r - \delta \right\} = 0.
\]

For each fixed \( b \) there is a solution \( r(b) \). For large \( x \) we see that we have to maximise \( r(b) \).

If now \( c(b) \), \( \mathbb{E}[R(Y, b)] \), and \( \mathbb{E}[R^2(Y, b)] \) are continuous functions then there is \( b \in [0, a] \) where \( r(b) \) is maximised.
The principle of smooth fit suggest, that \( V(x) = e^{-rx}/r \).

Using martingale techniques we can prove that \( V(x) \) really is the minimal value of the strategies.
Suppose $R(Y, b) = bY$ and premia are calculated via an expected value principle. Then we obtain the equation

$$\inf_{b \in [0,1]} \left\{ \frac{1}{2} \lambda b^2 \mu_2 r^2 - [\eta b - (\eta - \theta)] \lambda \mu r - \delta \right\} = 0$$

with $\eta > \theta$.

We find

$$b = \frac{\eta \mu}{\mu_2 r}, \quad r = \frac{1}{\eta - \theta} \left[ \frac{\delta}{\lambda \mu} + \frac{\eta^2 \mu}{2 \mu_2} \right],$$

provided $b \leq 1$. Otherwise $b = 1$ and $r$ is the corresponding solution.
The Classical Model

For a classical model we obtain

\[ X^b_t = x + \int_0^t c(b_s) \, ds - \sum_{k=1}^{N_t} R(Y_k, b_{T_k}) + Z_t, \]

where \( T_k \) is the time of the \( k \)-th claim.

We assume:

- \( c(b) \) is increasing and continuous.
- \( c(0) < 0, c(a) = c \).
- \( R(y, b) \) is increasing and continuous in \( b \).
The Gerber–Shiu Function

Let \( \tau^b = \inf \{ t : X^b_t - Z^b_t < 0 \} \). Then we can express the value function as

\[
V^b(x) = \mathbb{E}[e^{-\delta \tau^b}(V(0) + Z^b_{\tau^b})].
\]

This is the Gerber–Shiu function for the process without injections. That is, we look for a strategy minimising the Gerber–Shiu function.
The Hamilton–Jacobi–Bellman Equation

From \( V(x) \leq y + V(x + y) \) we conclude that \( V'(x) \geq -1 \).
We can show that the minimal value fulfills

\[
\inf_{0 \leq b \leq a} c(b) V'(x) + \lambda \left[ \int_{0}^{\infty} V(x - R(y, b)) \, dG(y) - V(x) \right] - \delta V(x) = 0,
\]

where \( V(z) = V(0) - z \) for \( z < 0 \).
The Verification Theorem

Suppose we have a decreasing solution $f(x)$ to the Hamilton–Jacobi–Bellman equation vanishing in infinity. Let $b(x)$ be the argument at which the inf is taken. If $c(b(0)) \leq 0$ we need the correct initial value

$$V(0) = \frac{(\lambda + \delta) \mathbb{E}[R(Y, b(0))] - c(b(0))}{\delta}.$$ 

Then $V(x) = f(x)$ and $\{b(X_t)\}$ is an optimal strategy. Note that because $c(b)$ and $R(y, b)$ are continuous in $b$ an optimal $b(x)$ exists.
Reformulation of the Problem

We can write

\[ V'(x) = \sup_{0 < b \leq a} \frac{(\lambda + \delta)V(x) - \lambda \int_0^\infty V(x - R(y, b)) \, dG(y)}{c(b)} . \]
Construction of a Solution

Let $V_0(x)$ be the value function for the strategy $b_t = a$. Define recursively for $x \geq 0$

$$V_{n+1}(x) = \int_0^\infty \inf_{0 < b \leq a} \frac{\lambda \int_0^\infty V_n(v - R(y, b)) \, dG(y) - (\lambda + \delta) V_n(v)}{c(b)} \, dv,$$

and $V_{n+1}(z) = V_{n+1}(0) - z$ for $z < 0$.

The value function $V(x)$ is a fixed point of this equation.
Proportional Reinsurance: Convexity

For proportional reinsurance with premia calculated via an expected value principle we get the process

$$X_t^b = x + \int_0^t [b_s(1 + \eta) - (\eta - \theta)] \, ds - \sum_{k=1}^{N_t} b_{T_k} - Y_k - Z_t.$$  

Let $b^x$ and $b^y$ be a strategy for initial capital $x$ and $y$, respectively. Then the strategy $\alpha b^x + \beta b^y$ for initial capital $z = \alpha x + \beta y$, where $\beta = 1 - \alpha$, has the property that $Z^z_t \leq \alpha Z^x_t + \beta Z^y_t$. Thus $V(\alpha x + \beta y) \leq \alpha V(x) + \beta V(y)$; i.e., $V(x)$ is convex.
Further Problems

- Allow $\delta < 0$; i.e., capital injection today is preferred to capital injection tomorrow.
- A fixed penalty has to be paid at any time a capital injection is done.
- The investors can choose between a capital injection, or paying a fixed penalty and stopping the business.
- The surplus can be invested into a risky asset.


References


