Model risk in claims reserving within Tweedie's compound Poisson models

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Gareth Peters (UNSW/CSIRO), Pavel Shevchenko (CSIRO), and Mario Wüthrich (ETH). ***Model risk in claims reserving within Tweedie's compound Poisson models,*** **preprint 2008.**

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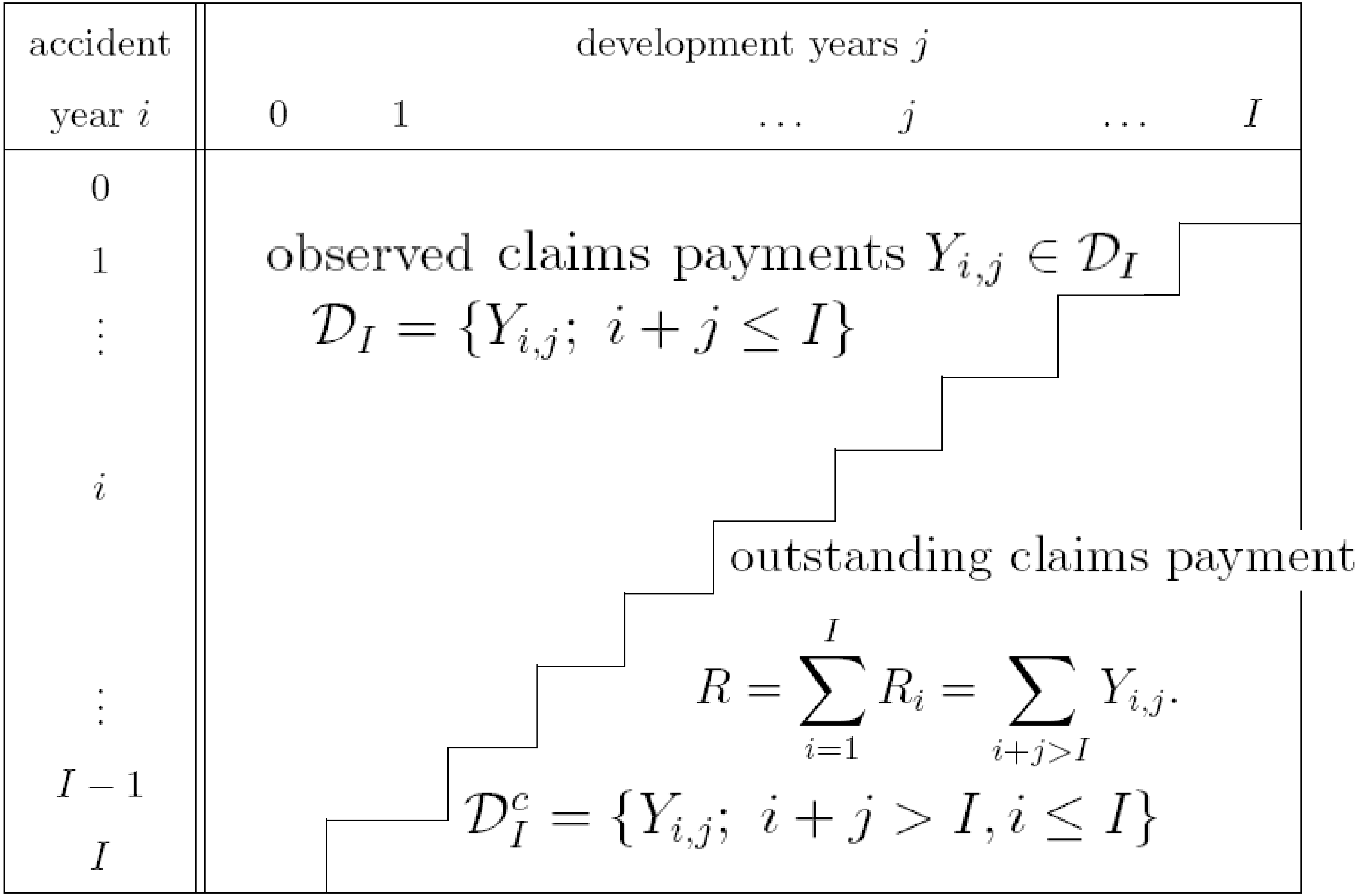
Claims Reserving (non-life insurance), solvency requirements, claims development triangle



Year	0	1	2	3	4	5	6	7	8	9
0	594.6975	372.1236	89.5717	20.7760	20.6704	6.2124	6.5813	1.4850	1.1130	1.5813
1	634.6756	324.6406	72.3222	15.1797	6.7824	3.6603	5.2752	1.1186	1.1646	
2	626.9090	297.6223	84.7053	26.2768	15.2703	6.5444	5.3545	0.8924		
3	586.3015	268.3224	72.2532	19.0653	13.2976	8.8340	4.3329			
4	577.8885	274.5229	65.3894	27.3395	23.0288	10.5224				
5	618.4793	282.8338	57.2765	24.4899	10.4957					
6	560.0184	289.3207	56.3114	22.5517						
7	528.8066	244.0103	52.8043							
8	529.0793	235.7936								
9	567.5568									

Table 2: Data - annual claims Y_{ij} for each accident year i and development year j

Content

- Tweedie's compound Poisson family to model annual claims
- Process Uncertainty, Parameter Estimation Error, Model uncertainty
- Variable selection
- Maximum likelihood and Bayesian estimation
- MCMC (random walk Metropolis-Hastings within Gibbs)
- Analysis/Conclusions

$\widehat{R}$ - predictor for R and estimator for $E\left[R \mid \mathcal{D}_I\right]$

$$R = \sum_{i=1}^{I} R_i = \sum_{i+j>I} Y_{i,j} \qquad E\left[R \mid \mathcal{D}_I\right] = \sum_{i=1}^{I} E\left[R_i \mid \mathcal{D}_I\right]$$

$$\mathrm{msep}_{R|\mathcal{D}_I}\left(\widehat{R}\right) = E\left[\left(R - \widehat{R}\right)^2 \middle| \mathcal{D}_I\right]$$

Mean Square Error of Prediction

$$\mathrm{msep}_{R|\mathcal{D}_I}\left(\widehat{R}\right) = \mathrm{Var}\left(R \mid \mathcal{D}_I\right) + \left(E\left[R \mid \mathcal{D}_I\right] - \widehat{R}\right)^2$$

$$= \text{process variance} + \text{estimation error}$$

$\widehat{R} = E\left[R \mid \mathcal{D}_I\right]$ *"best estimate" of reserve*

Bayesian context – variance decomposition

$$\mathrm{Var}\left(R \mid \mathcal{D}_I\right) = E\left[\mathrm{Var}\left(R \mid \boldsymbol{\theta}, \mathcal{D}_I\right) \mid \mathcal{D}_I\right] + \mathrm{Var}\left(E\left[R \mid \boldsymbol{\theta}, \mathcal{D}_I\right] \mid \mathcal{D}_I\right)$$

$$= \text{average process variance} + \text{parameter estimation error.}$$

$\boldsymbol{\theta}$ is model parameter vector modelled as random variable

Tweedie's compound Poisson model

$$Y_{i,j} \text{ are independent for } i,j \in \{0,\ldots,I\}$$

$$Y_{i,j} = 1_{\{N_{i,j}>0\}} \sum_{k=1}^{N_{i,j}} X_{i,j}^{(k)}$$

$$N_{i,j} \text{ and } X_{i,j}^{(k)} \text{ are independent for all } k$$

$$N_{i,j} \text{ is Poisson distributed with parameter } \lambda_{i,j}$$

$$X_{i,j}^{(k)} \text{ are independent gamma severities with}$$
$$\text{mean } \tau_{i,j} > 0 \text{ and shape parameter } \gamma > 0$$

Tweedie's compound Poisson: exponential dispersion family representation

$$P\left[Y_{i,j}=0\right]=P\left[N_{i,j}=0\right]=\exp\left\{-\phi_{i,j}^{-1}\kappa_p(\theta_{i,j})\right\}$$

$$f_{\theta_{i,j}}(y;\phi_{i,j},p)=c(y;\phi_{i,j},p)\exp\left\{\frac{y\,\theta_{i,j}-\kappa_p(\theta_{i,j})}{\phi_{i,j}}\right\}\ \text{ for } y>0$$

where $\quad \theta_{i,j}<0,\ \phi_{i,j}>0,\quad \kappa_p(\theta)\overset{def.}{=}\dfrac{1}{2-p}\left[(1-p)\theta\right]^{\gamma}$

$$c(y;\phi,p)=\sum_{r\geq 1}\left(\frac{(1/\phi)^{\gamma+1}y^{\gamma}}{(p-1)^{\gamma}(2-p)}\right)^{r}\frac{1}{r!\Gamma(r\gamma)y}$$

$$p=p(\gamma)=\frac{\gamma+2}{\gamma+1}\in(1,2)\quad \phi_{i,j}=\frac{\lambda_{i,j}^{1-p}\tau_{i,j}^{2-p}}{2-p}>0$$

$$\theta_{i,j}=\left(\frac{1}{1-p}\right)(\mu_{i,j})^{(1-p)}<0,\ \ \mu_{i,j}=\lambda_{i,j}\tau_{i,j}>0$$

Final representation: Tweedie's compound Poisson model

$$P\left[Y_{i,j}=0\right]=P\left[N_{i,j}=0\right]=\exp\left\{-\phi_{i,j}^{-1}\frac{\mu_{i,j}^{2-p}}{2-p}\right\}$$

$$f_{\mu_{i,j}}(y;\phi_{i,j},p)=c(y;\phi_{i,j},p)\exp\left\{\phi_{i,j}^{-1}\left[y\,\frac{\mu_{i,j}^{1-p}}{1-p}-\frac{\mu_{i,j}^{2-p}}{2-p}\right]\right\}\quad \text{for } y>0$$

$$E\left[Y_{i,j}\right]=\frac{\partial}{\partial\theta_{i,j}}\kappa_p(\theta_{i,j})=\kappa_p'(\theta_{i,j})=\left[(1-p)\theta_{i,j}\right]^{1/(1-p)}=\mu_{i,j}$$

$$\operatorname{Var}\left(Y_{i,j}\right)=\phi_{i,j}\kappa_p''(\theta_{i,j})=\phi_{i,j}\;\mu_{i,j}^{p}$$

$p\in(1,2)$, typically fixed by the modeller *Model Risk*

$p\to 1$, overdispersed Poisson model

$p\to 2$, gamma model

$p=0$, Gaussian density

$p=3$, inverse Gaussian

Parameter estimation

estimate $\mu_{i,j}$, p and $\phi_{i,j}$ using observations $\mathcal{D}_I$

Model Assumptions

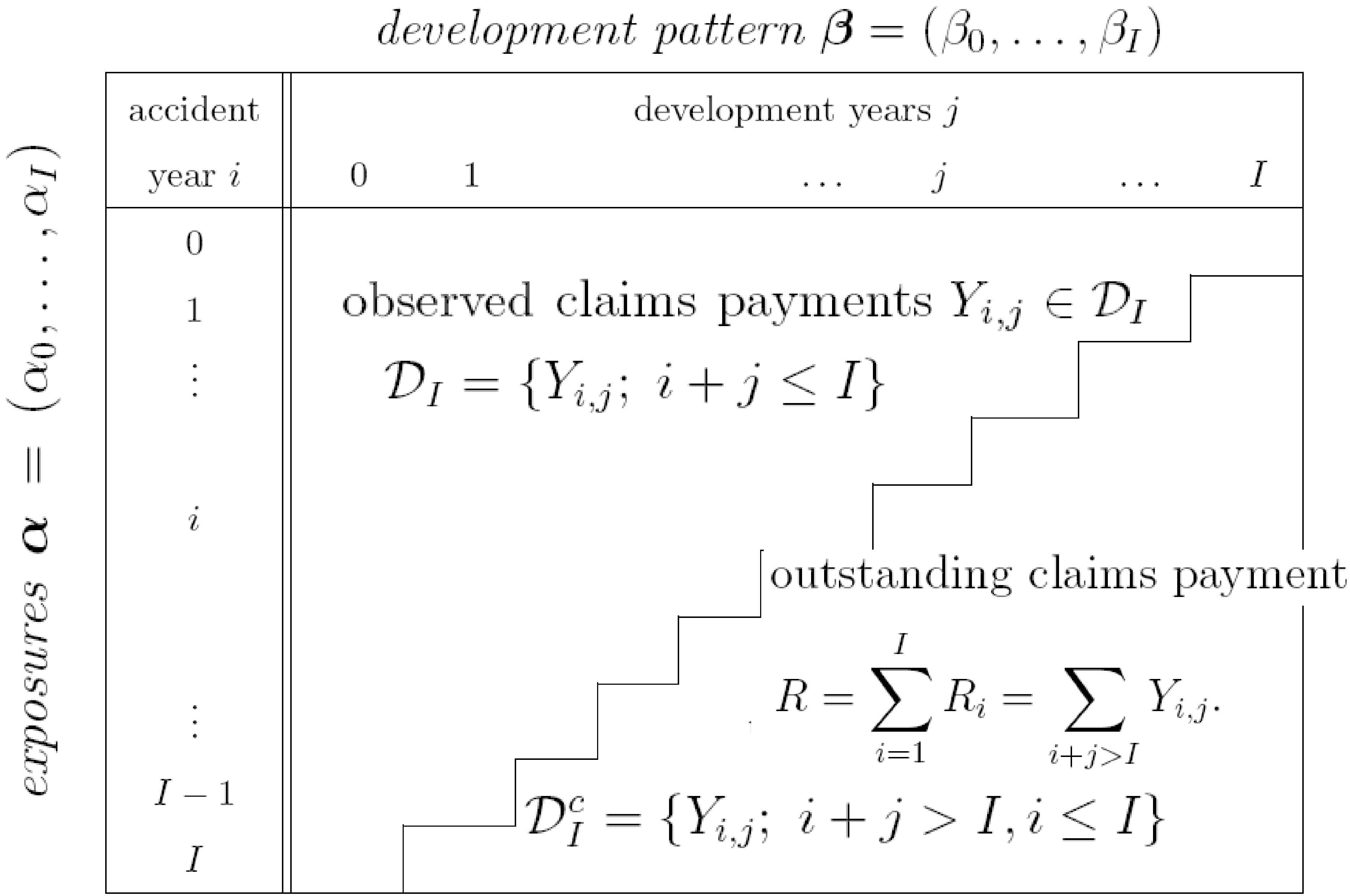
multiplicative model $\quad \mu_{i,j} = \alpha_i \; \beta_j$

exposures $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_I)$

development pattern $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_I)$

$\phi_{i,j} = \phi$ *and* $\alpha_i > 0,\ \beta_j > 0$

$\alpha_0 = 1$


development pattern $\boldsymbol{\beta} = (\beta_0, \ldots, \beta_I)$
exposures $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_I)$
accident year i
development years j
0 1 … j … I
0
1
⋮
i
⋮
$I-1$
I
observed claims payments $Y_{i,j} \in \mathcal{D}_I$
$\mathcal{D}_I = \{Y_{i,j};\ i+j \le I\}$
outstanding claims payment
$R = \sum_{i=1}^{I} R_i = \sum_{i+j>I} Y_{i,j}.$
$\mathcal{D}_I^c = \{Y_{i,j};\ i+j > I, i \le I\}$

Likelihood function

$$\boldsymbol{\theta} = (p, \phi, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

$$L_{\mathcal{D}_I}(\boldsymbol{\theta}) = \prod_{i+j \le I} c(Y_{i,j}; \phi, p) \exp\left\{ \phi^{-1} \left[Y_{i,j} \frac{(\alpha_i \beta_j)^{1-p}}{1-p} - \frac{(\alpha_i \beta_j)^{2-p}}{2-p} \right] \right\}$$

$$c(y; \phi, p) = \sum_{r \ge 1} \left(\frac{(1/\phi)^{\gamma+1} y^{\gamma}}{(p-1)^{\gamma}(2-p)} \right)^r \frac{1}{r!\Gamma(r\gamma)y} = \frac{1}{y} \sum_{r \ge 1} W_r$$

$$\log W_r = r \log z - \log \Gamma(1+r) - \log \Gamma(\gamma r)$$

$$z = \frac{(1/\phi)^{\gamma+1} y^{\gamma}}{(p-1)^{\gamma}(2-p)}$$

$$\log W_r \approx r \{ \log z + (1+\gamma) - \gamma \log \gamma - (1+\gamma) \log r \} - \log(2\pi) - \frac{1}{2} \log \gamma - \log r$$

Likelihood evaluation, Dunn and Smyth (2005)

$$\frac{\partial \log W_r}{\partial r} \approx \log z - \log r - \gamma \log(\gamma r) \qquad W_r \text{ is unimodal in } r$$

Solving $\partial W_r / \partial r = 0$ results in $R_0 = R_0(\phi, p) = \dfrac{y^{2-p}}{(2-p)\phi}$

find $R_L < R_0 < R_U$ such that

$$c(y;\phi,p) \approx \tilde{c}(y;\phi,p) = \frac{1}{y}\sum_{r=R_L}^{R_U} W_r$$

$$c(y;\phi,p) - \tilde{c}(y;\phi,p) < W_{R_L-1}\frac{1-q_L^{R_L-1}}{1-q_L} + W_{R_U+1}\frac{1}{1-q_U}$$

$$q_L = \exp\left(\frac{\partial \log W_r}{\partial r}\right)\Bigg|_{r=R_L-1}, \qquad q_U = \exp\left(\frac{\partial \log W_r}{\partial r}\right)\Bigg|_{r=R_U+1}$$

Likelihood evaluation numerically

find $R_L < R_0 < R_U$ such that

$W_{R_L} \leqslant e^{-37} W_{R_0}$ (or $R_L = 1$) and $W_{R_U} \leqslant e^{-37} W_{R_0}$

to avoid numerical overflow problems

$$c(y;\phi,p) \approx \widetilde{c}(y;\phi,p) = \frac{1}{y} \sum_{r=R_L}^{R_U} W_r$$

$$\log \widetilde{c}(y;\phi,p) = -\log y + \log W_{R_0} + \log\left(\sum_{r=R_L}^{R_U} \exp\left(\log\left(W_R\right) - \log\left(W_{R_0}\right)\right)\right)$$

if p is close to 1, then likelihood may become multimodal
if p is close to 2, then number of terms to evaluate may become large.
For our dataset, we restrict p to be in the range [1.1; 1.95]

Maximum likelihood estimation

maximizing $L_{\mathcal{D}_I}(\boldsymbol{\theta})$ in $\boldsymbol{\theta} = (p, \phi, \boldsymbol{\alpha}, \boldsymbol{\beta})$ leads to

$$\widehat{\boldsymbol{\theta}}^{\mathrm{MLE}} = (\widehat{p}^{\mathrm{MLE}}, \widehat{\phi}^{\mathrm{MLE}}, \widehat{\boldsymbol{\alpha}}^{\mathrm{MLE}}, \widehat{\boldsymbol{\beta}}^{\mathrm{MLE}})$$

typically MLE is done for fixed p (expert choice)

$$\widehat{R}^{\mathrm{MLE}} = \sum_{i+j>I} \widehat{\alpha}_i^{\mathrm{MLE}} \, \widehat{\beta}_j^{\mathrm{MLE}} \quad \text{the best estimate reserves for } R$$

$$\frac{\partial \ln L_{\mathcal{D}_I}(\boldsymbol{\theta})}{\partial \beta_k} = \frac{\partial}{\partial \beta_k} \sum_{j=0}^{I} \sum_{i=0}^{I-j} \phi^{-1} \left(Y_{i,j} \frac{(\alpha_i \beta_j)^{1-p}}{1-p} - \frac{(\alpha_i \beta_j)^{2-p}}{2-p} \right)$$

$$= \sum_{i=0}^{I-k} \phi^{-1} \left(Y_{i,k} \alpha_i^{1-p} \beta_k^{-p} - \alpha_i^{2-p} \beta_k^{1-p} \right)$$

$$\beta_k = \frac{\sum_{i=0}^{I-k} Y_{i,k} \alpha_i^{1-p}}{\sum_{i=0}^{I-k} \alpha_i^{2-p}}, \qquad k = 0, \ldots, I$$

Covariances between maximum likelihood estimators

Fisher Information matrix

$$(\boldsymbol{I})_{i,j} = -\left.\frac{\partial^2 \ln L_{\mathcal{D}_I}(\boldsymbol{\theta})}{\partial\theta_i \partial\theta_j}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}^{\mathrm{MLE}}}$$

$$\mathrm{cov}\left(\widehat{\theta}_i^{\mathrm{MLE}}, \widehat{\theta}_j^{\mathrm{MLE}}\right) = \left(\boldsymbol{I}^{-1}\right)_{i,j}$$

$$\widehat{\beta}_I^{\mathrm{MLE}} = Y_{0,I}$$

$$\mathrm{cov}(\widehat{\beta}_I^{\mathrm{MLE}}, \widehat{\theta}_i^{\mathrm{MLE}}) = 0, \quad \widehat{\theta}_i^{\mathrm{MLE}} \neq \widehat{\beta}_I^{\mathrm{MLE}}$$

Maximum likelihood: process and estimation errors

$$\widehat{R}^{\mathrm{MLE}} = \sum_{i+j>I} \widehat{\alpha}_i^{\mathrm{MLE}} \; \widehat{\beta}_j^{\mathrm{MLE}}$$

$$\mathrm{stdev}\left(\widehat{R}^{\mathrm{MLE}}\right) = \sqrt{\mathrm{Var}\left(\widehat{R}^{\mathrm{MLE}}\right)} \quad \text{parameter estimation error}$$

$$\begin{aligned}
\widehat{\mathrm{Var}}\left(\widehat{R}^{\mathrm{MLE}}\right) = & \sum_{i_1+j_1>I} \sum_{i_2+j_2>I} \widehat{\alpha}_{i_1}^{\mathrm{MLE}} \widehat{\alpha}_{i_2}^{\mathrm{MLE}} \mathrm{cov}\left(\widehat{\beta}_{j_1}^{\mathrm{MLE}}, \widehat{\beta}_{j_2}^{\mathrm{MLE}}\right) \\
& + \sum_{i_1+j_1>I} \sum_{i_2+j_2>I} \widehat{\beta}_{j_1}^{\mathrm{MLE}} \widehat{\beta}_{j_2}^{\mathrm{MLE}} \mathrm{cov}\left(\widehat{\alpha}_{i_1}^{\mathrm{MLE}}, \widehat{\alpha}_{i_2}^{\mathrm{MLE}}\right) \\
& + 2 \sum_{i_1+j_1>I} \sum_{i_2+j_2>I} \widehat{\alpha}_{i_1}^{\mathrm{MLE}} \widehat{\beta}_{j_2}^{\mathrm{MLE}} \mathrm{cov}\left(\widehat{\alpha}_{i_2}^{\mathrm{MLE}}, \widehat{\beta}_{j_1}^{\mathrm{MLE}}\right)
\end{aligned}$$

$$\widehat{\mathrm{Var}}(R) = \sum_{i+j>I} \left(\widehat{\alpha}_i^{\mathrm{MLE}} \; \widehat{\beta}_j^{\mathrm{MLE}}\right)^{\widehat{p}^{\mathrm{MLE}}} \widehat{\phi}^{\mathrm{MLE}} \quad \text{process variance}$$

$$\begin{aligned}
\widehat{\mathrm{msep}}_{R|\mathcal{D}_I}\left(\widehat{R}^{\mathrm{MLE}}\right) &= \widehat{\mathrm{Var}}(R) + \widehat{\mathrm{Var}}\left(\widehat{R}^{\mathrm{MLE}}\right) \\
&= \text{MLE process variance} + \text{MLE estimation error}
\end{aligned}$$

Bayesian inference

$\boldsymbol{\theta} = (p, \phi, \boldsymbol{\alpha}, \boldsymbol{\beta})$ are treated as random

$\pi(\boldsymbol{\theta} \mid \mathcal{D}_I) \propto L_{\mathcal{D}_I}(\boldsymbol{\theta})\ \pi(\boldsymbol{\theta})$ Markov Chain Monte Carlo (MCMC)

$$MAP: \qquad \hat{\boldsymbol{\theta}}^{MAP} = \arg\max_{\boldsymbol{\theta}} \left[\pi(\boldsymbol{\theta} \mid \mathcal{D}_I)\right],$$

$$MMSE: \qquad \hat{\boldsymbol{\theta}}^{MMSE} = E\left[\boldsymbol{\theta} \mid \mathcal{D}_I\right].$$

if the prior $\pi(\boldsymbol{\theta})$ is constant and the parameter range includes the MLE, then the MAP of the posterior is the same as the MLE.

Gaussian approximation for $\pi(\boldsymbol{\theta} \mid \mathcal{D}_I)$

$$\ln \pi(\boldsymbol{\theta} \mid \mathcal{D}_I) \approx \ln \pi(\hat{\boldsymbol{\theta}}^{MAP} \mid \mathcal{D}_I) + \frac{\partial^2}{\partial\theta_i \partial\theta_j} \ln \pi(\boldsymbol{\theta} \mid \mathcal{D}_I)\Big|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}^{MAP}} \left(\theta_i - \hat{\theta}_i^{MAP}\right)\left(\theta_j - \hat{\theta}_j^{MAP}\right)$$

Bayesian inference estimates

$$E\left[R\middle|\mathcal{D}_I\right] = \sum_{i+j>I} E\left[\alpha_i\beta_j\middle|\mathcal{D}_I\right] \qquad \widetilde{R} = E\left[R\middle|\boldsymbol{\theta}\right] = \sum_{i+j>I} \alpha_i\beta_j$$

the best consistent estimate of reserves (ER)

$$\widehat{R}^{\mathrm{B}} = E\left[\widetilde{R}\,\middle|\,\mathcal{D}_I\right] = \sum_{i+j>I} E\left[\alpha_i\beta_j\middle|\mathcal{D}_I\right] = E\left[R\middle|\mathcal{D}_I\right]$$

$$\mathrm{msep}_{R|\mathcal{D}_I}\left(\widehat{R}^{\mathrm{B}}\right) = E\left[\left(R-\widehat{R}^{\mathrm{B}}\right)^2\middle|\mathcal{D}_I\right] = \mathrm{Var}\left(R\middle|\mathcal{D}_I\right)$$

$$\mathrm{Var}\left(R\middle|\mathcal{D}_I\right) = \mathrm{Var}\left(\sum_{i+j>I} Y_{i,j}\middle|\mathcal{D}_I\right)$$

$$= \sum_{i+j>I} E\left[(\alpha_i\beta_j)^p\,\phi\middle|\mathcal{D}_I\right] + \mathrm{Var}\left(\widetilde{R}\,\middle|\,\mathcal{D}_I\right)$$

= average process variance (PV) + parameter estimation error (EE)

Note, model error is incorporated via averaging over values of p

Random Walk Metropolis Hastings (RW-MH) within Gibbs

1. Initialize randomly or deterministically for $t = 0$ the parameter vector $\boldsymbol{\theta}^{t=0}$ to the maximum likelihood estimates.

2. For $t = 1, ..., T$

a) Set $\boldsymbol{\theta}^{t} = \boldsymbol{\theta}^{t-1}$

b) For $i = 1, ..., 2I+3$

Sample proposal θ_i^* from Gaussian truncated density

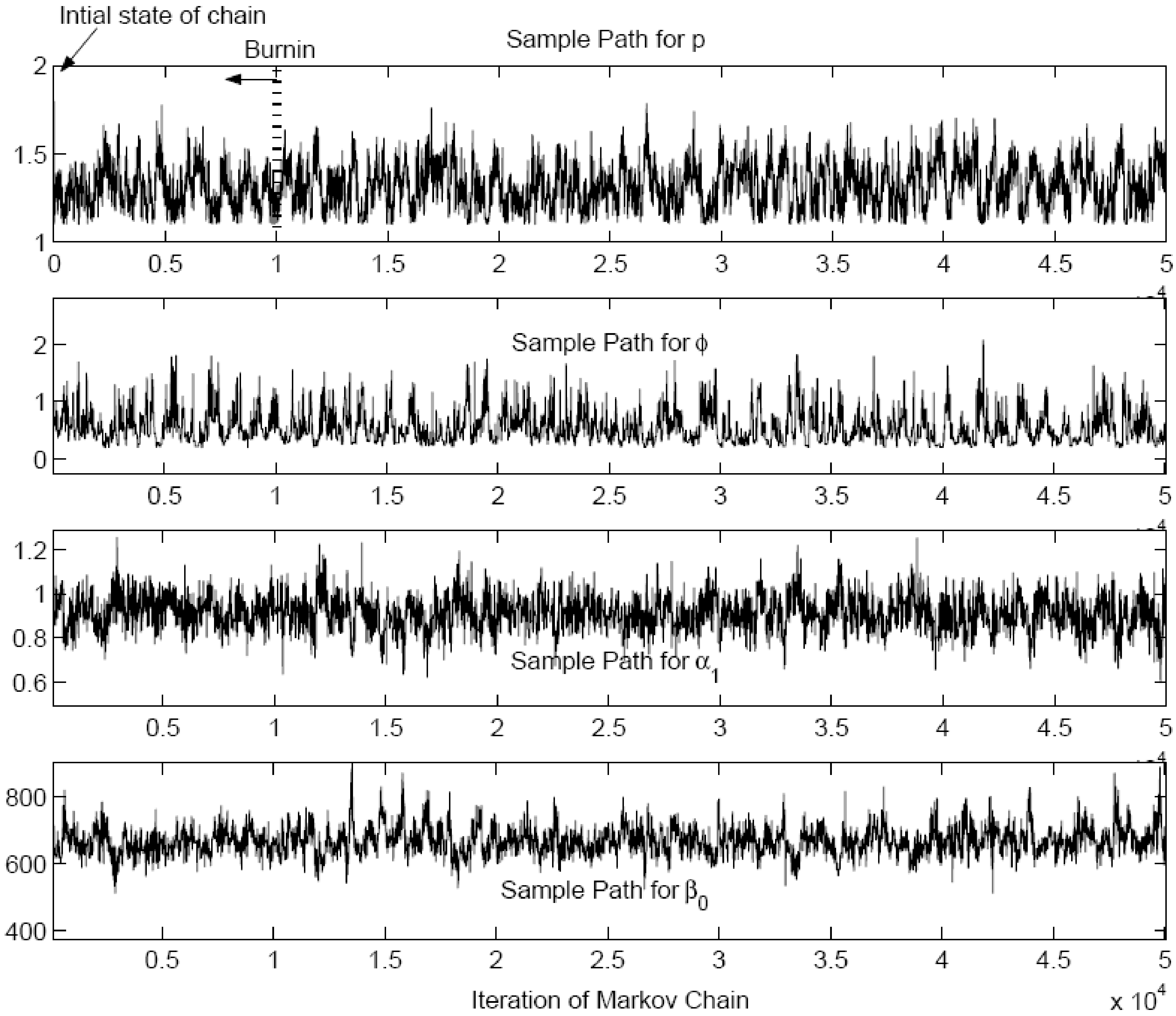
$$f_N^T\left(\theta_i^*; \theta_i^t, \sigma_{RWi}\right) = \frac{f_N\left(\theta_i^*; \theta_i^t, \sigma_{RWi}\right)}{F_N\left(b_i; \theta_i^t, \sigma_{RWi}\right) - F_N\left(a_i; \theta_i^t, \sigma_{RWi}\right)}$$

to obtain $\boldsymbol{\theta}^* = \left(\theta_1^t, \ldots, \theta_{i-1}^t, \theta_i^*, \theta_{i+1}^{t-1}, \ldots\right)$

Accept proposal with acceptance probability

$$\alpha\left(\boldsymbol{\theta}^t, \boldsymbol{\theta}^*\right) = \min\left\{1, \frac{\pi(\boldsymbol{\theta}^* \mid \mathcal{D}_I) f_N^T\left(\theta_i^t; \theta_i^*, \sigma_{RWi}\right)}{\pi(\boldsymbol{\theta}^t \mid \mathcal{D}_I) f_N^T\left(\theta_i^*; \theta_i^t, \sigma_{RWi}\right)}\right\}$$

***Note**: normalization constant in posterior is not needed; optimal acceptance rate is 0.234*


Intial state of chain
Burnin
Sample Path for p
Sample Path for ϕ
Sample Path for α1
Sample Path for β0
Iteration of Markov Chain
x 10^4

The prior domains

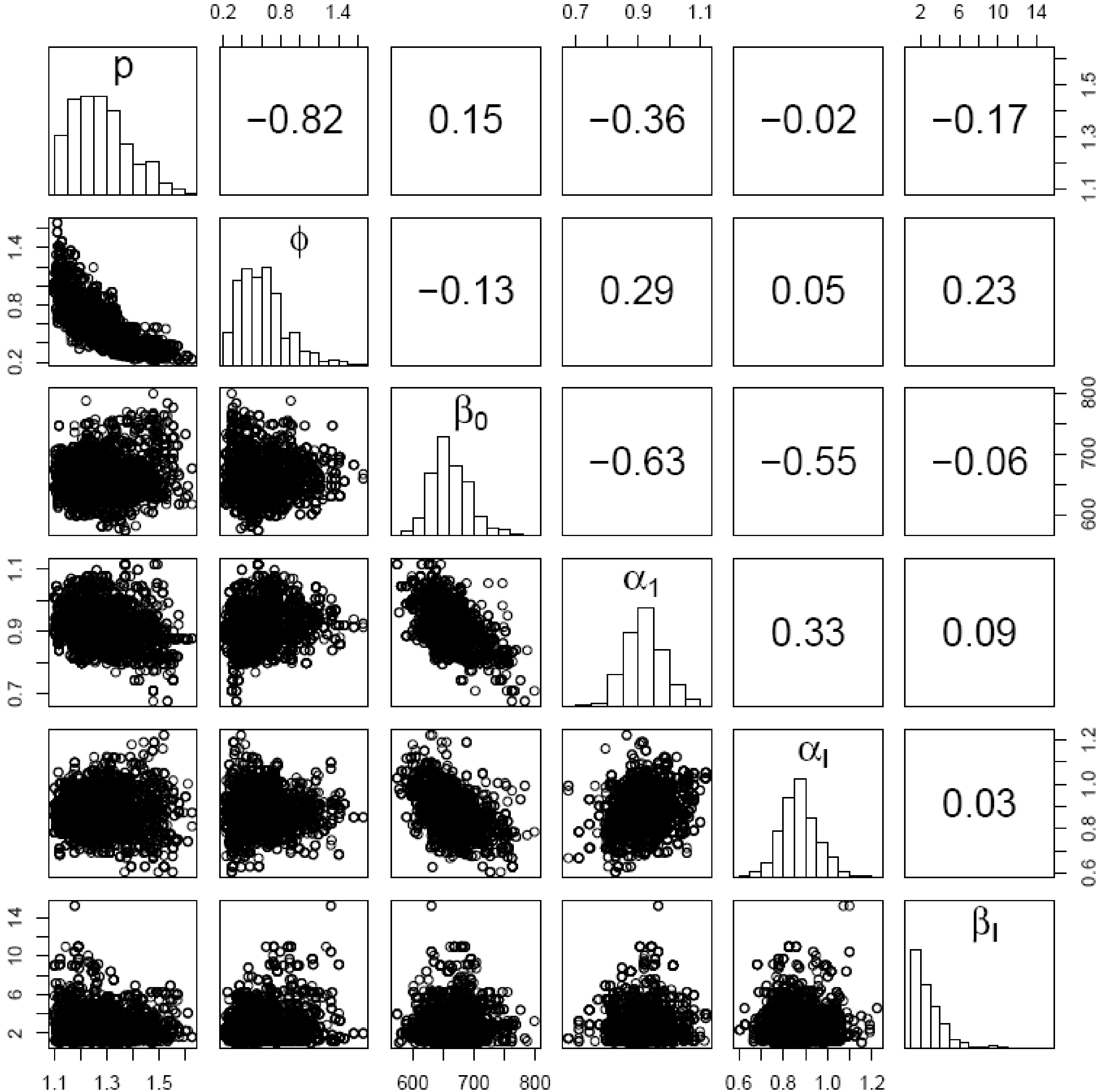
$p \in (1.1, 1.95)$, $\phi \in (0.01, 100)$, $\alpha_i \in (0.01, 100)$ and $\beta_j \in (0.01, 10^4)$

	MLE	**MLE stdev**	**Bayesian posterior**			σ_{RW}
			MMSE	**stdev**	$[Q_{0.05}; Q_{0.95}]$	
p	1.259	0.149	1.332 (0.007)	0.143 (0.004)	[1.127;1.590]	1.61
ϕ	0.351	0.201	0.533 (0.013)	0.289 (0.005)	[0.174;1.119]	1.94
α_1	0.918	0.056	0.901 (0.004)	0.074 (0.001)	[0.778;1.022]	0.842
α_2	0.946	0.051	0.946 (0.003)	0.073 (0.001)	[0.833;1.072]	0.907
α_3	0.861	0.048	0.861 (0.003)	0.068 (0.001)	[0.756;0.977]	0.849
α_4	0.891	0.049	0.902 (0.003)	0.072 (0.002)	[0.794;1.027]	0.893
α_5	0.879	0.051	0.876 (0.003)	0.070 (0.001)	[0.768;0.994]	0.932
α_6	0.842	0.048	0.843 (0.002)	0.069 (0.001)	[0.736;0.958]	0.751
α_7	0.762	0.046	0.762 (0.003)	0.066 (0.001)	[0.660;0.876]	0.888
α_8	0.763	0.047	0.765 (0.003)	0.067 (0.001)	[0.661;0.874]	0.897
α_9	0.848	0.059	0.856 (0.003)	0.090 (0.002)	[0.716;1.009]	1.276

Table 3: MLE and Bayesian estimators.

Table 3: MLE and Bayesian estimators.

	MLE	MLE stdev	Bayesian posterior			σ_{RW}
			MMSE	stdev	$[Q_{0.05}; Q_{0.95}]$	
β_0	669.1	27.7	672.7 (2.1)	39.7 (0.7)	[610.0;740.0]	296
β_1	329.0	14.4	331.1 (1.0)	20.6 (0.4)	[298.1;365.9]	190
β_2	77.43	4.38	78.06 (0.24)	6.10 (0.06)	[68.58;88.29]	75.4
β_3	24.59	1.96	24.95 (0.08)	2.64 (0.03)	[20.89;29.64]	40.9
β_4	16.28	1.55	16.65 (0.05)	2.09 (0.03)	[13.44;20.30]	40.6
β_5	7.773	1.028	8.068 (0.024)	1.356 (0.020)	[6.064;10.473]	26.0
β_6	5.776	0.937	6.115 (0.022)	1.261 (0.016)	[4.246;8.347]	24.1
β_7	1.219	0.396	1.494 (0.006)	0.609 (0.013)	[0.739;2.609]	13.1
β_8	1.188	0.476	1.622 (0.008)	0.802 (0.016)	[0.674;3.070]	15.1
β_9	1.581	0.790	2.439 (0.021)	1.496 (0.026)	[0.829;5.250]	32.1


p
φ
β0
α1
αI
βI
-0.82
0.15
-0.36
-0.02
-0.17
-0.13
0.29
0.05
0.23
-0.63
-0.55
-0.06
0.33
0.09
0.03

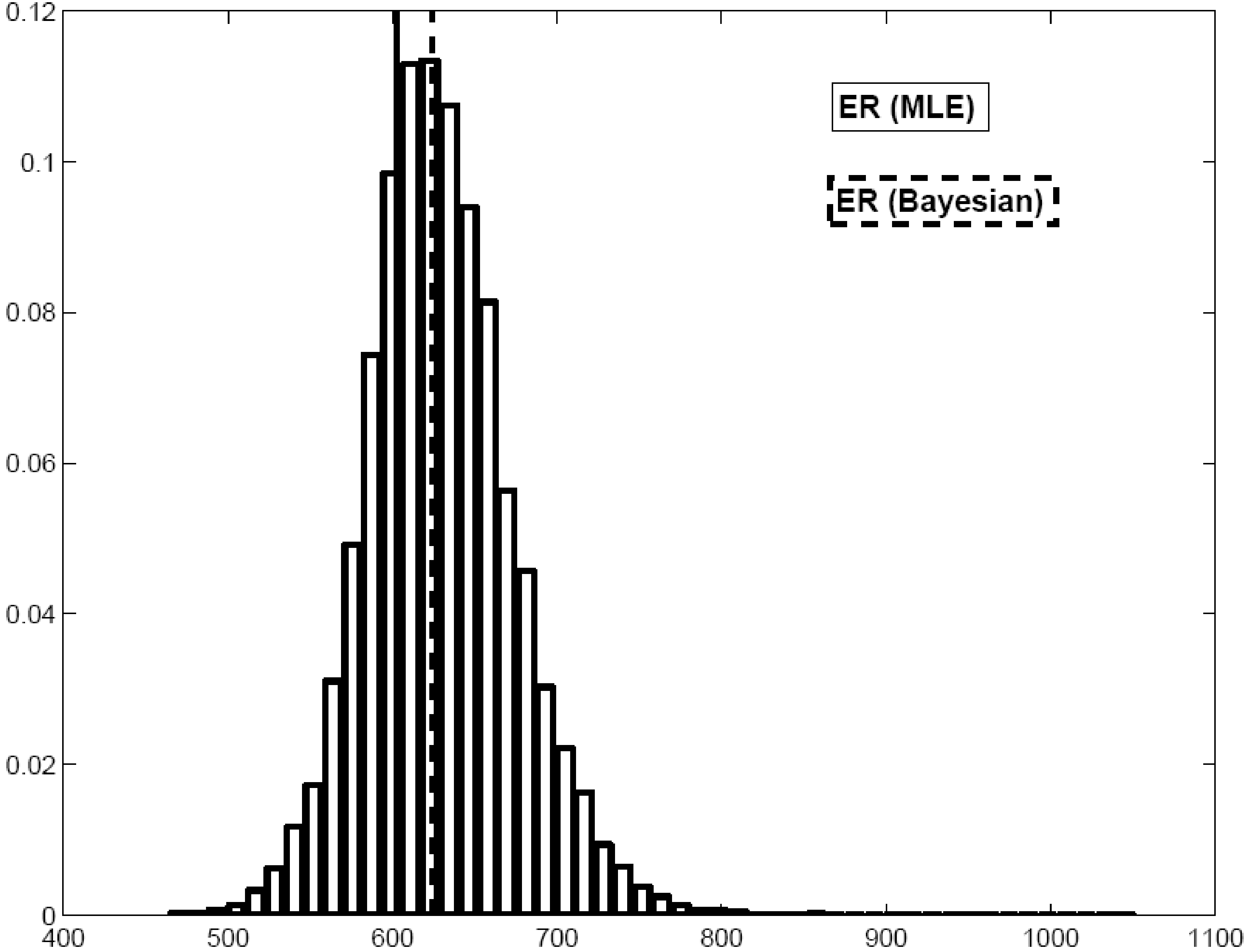


Figure 3: Predicted distribution of reserves, $\widetilde{R} = \sum_{i+j>I} \alpha_i \beta_j$.

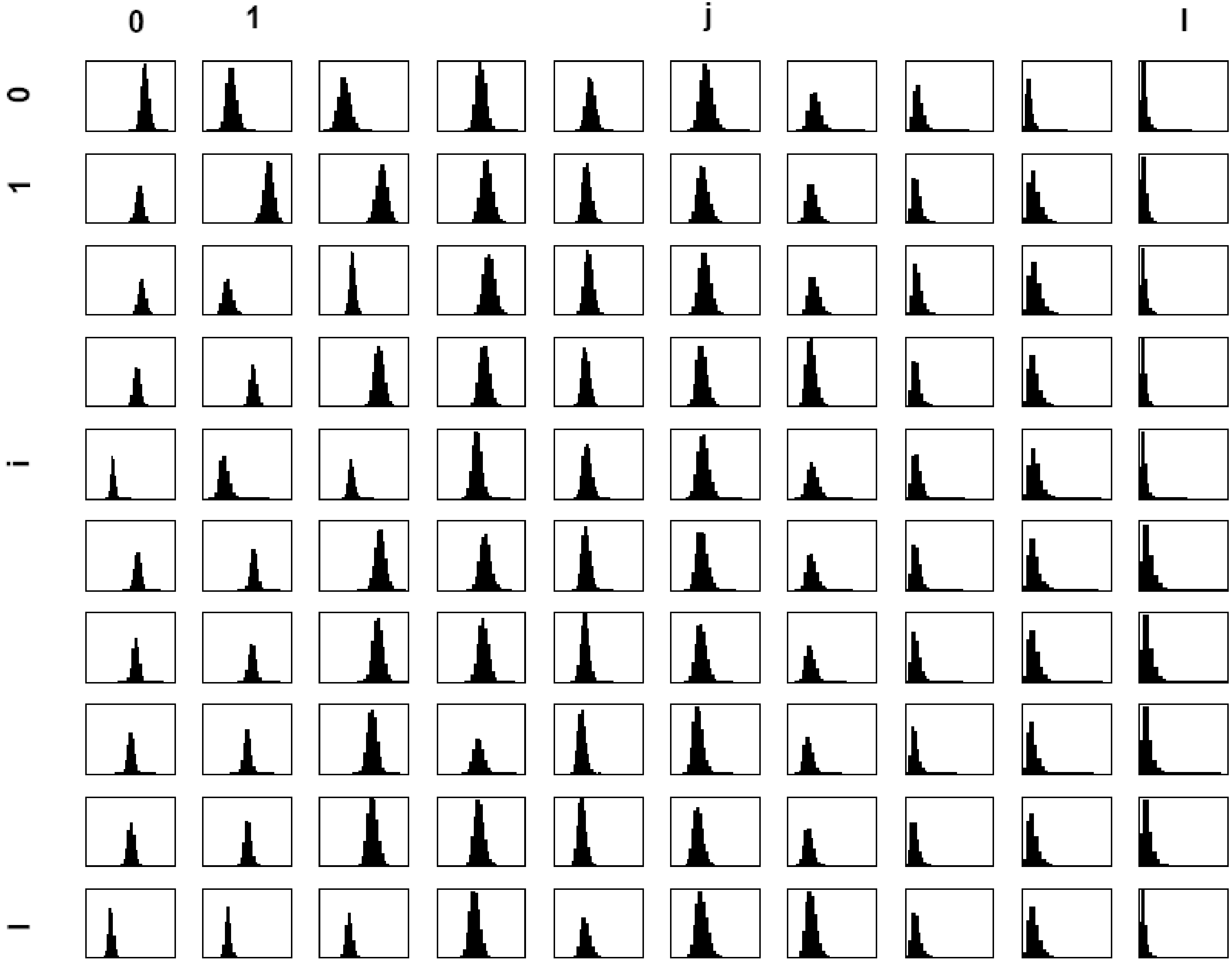


Figure 4: Posterior distributions for $\widetilde{R}_{i,j} = \alpha_i \beta_j$ estimated using MCMC.

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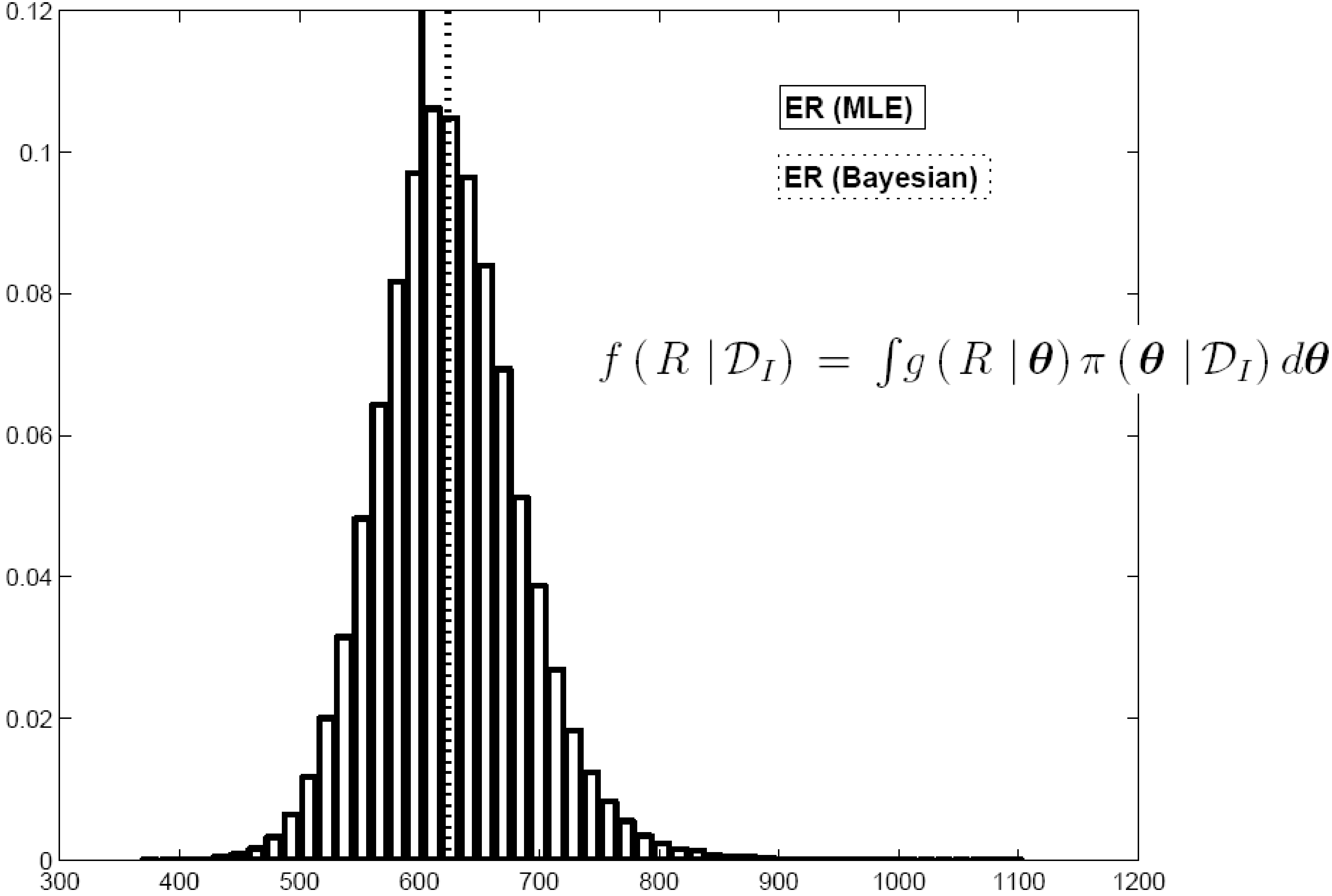
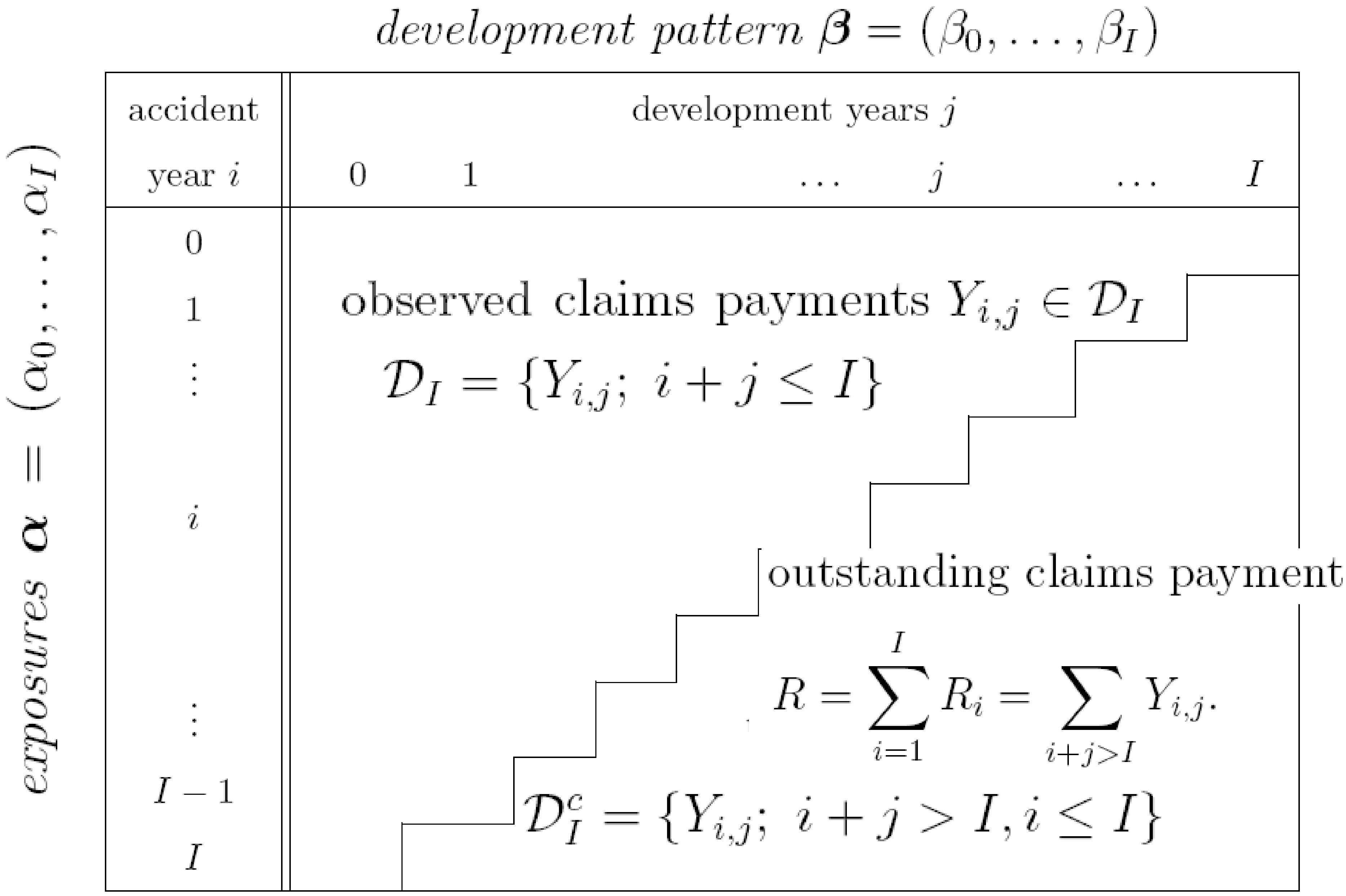


Figure 5: Distribution of total outstanding claims payment $R = \sum_{i+j>I} Y_{i,j}$, accounting for all process, estimation and model uncertainties.

Variable selection models



Variable selection models

- $M_0 : \boldsymbol{\theta}_{[0]} = \left(p, \phi, \widetilde{\alpha}_0 = \alpha_0, ..., \widetilde{\alpha}_I = \alpha_I, \widetilde{\beta}_0 = \beta_0, ..., \widetilde{\beta}_I = \beta_I\right)$ - **saturated model**.

- $M_1 : \boldsymbol{\theta}_{[1]} = \left(p, \phi, \widetilde{\beta}_0\right)$ with $\left(\widetilde{\beta}_0 = \beta_0 ... = \beta_I\right), (\alpha_0 = ... = \alpha_I = 1)$.

- $M_2 : \boldsymbol{\theta}_{[2]} = \left(p, \phi, \widetilde{\alpha}_1, \widetilde{\beta}_0, \widetilde{\beta}_1\right)$ with $(\alpha_0 = ... = \alpha_4 = 1)$, $(\widetilde{\alpha}_1 = \alpha_5 = ... = \alpha_I)$, $\left(\widetilde{\beta}_0 = \beta_0 = ... = \beta_4\right)$, $\left(\widetilde{\beta}_1 = \beta_5 = ... = \beta_I\right)$.

- $M_3 : \boldsymbol{\theta}_{[3]} = \left(p, \phi, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\beta}_0, \widetilde{\beta}_1, \widetilde{\beta}_2\right)$ with $(\alpha_0 = \alpha_1 = 1)$, $(\widetilde{\alpha}_1 = \alpha_2 = ... = \alpha_5)$, $(\widetilde{\alpha}_2 = \alpha_6 = ... = \alpha_I)$, $\left(\widetilde{\beta}_0 = \beta_0 = \beta_1\right)$, $\left(\widetilde{\beta}_1 = \beta_2 = ... = \beta_5\right)$, $\left(\widetilde{\beta}_2 = \beta_6 = ... = \beta_I\right)$

Variable selection models

- $M_4 : \boldsymbol{\theta}_{[4]} = \left(p, \phi, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3, \widetilde{\beta}_0, \widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3\right)$ with $(\alpha_0 = \alpha_1 = 1)$, $(\widetilde{\alpha}_1 = \alpha_2 = \alpha_3)$, $(\widetilde{\alpha}_2 = \alpha_4 = \alpha_5 = \alpha_6)$, $(\widetilde{\alpha}_3 = \alpha_7 = \alpha_8 = \alpha_I)$, $\left(\widetilde{\beta}_0 = \beta_0 = \beta_1\right)$, $\left(\widetilde{\beta}_1 = \beta_2 = \beta_3\right)$, $\left(\widetilde{\beta}_2 = \beta_4 = \beta_5 = \beta_6\right)$, $\left(\widetilde{\beta}_3 = \beta_7 = \beta_8 = \beta_I\right)$.

- $M_5 : \boldsymbol{\theta}_{[5]} = \left(p, \phi, \widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3, \widetilde{\alpha}_4, \widetilde{\beta}_0, \widetilde{\beta}_1, \widetilde{\beta}_2, \widetilde{\beta}_3, \widetilde{\beta}_4\right)$ with $(\alpha_0 = \alpha_1 = 1)$, $(\widetilde{\alpha}_1 = \alpha_2 = \alpha_3)$, $(\widetilde{\alpha}_2 = \alpha_4 = \alpha_5)$, $(\widetilde{\alpha}_3 = \alpha_6 = \alpha_7)$, $(\widetilde{\alpha}_4 = \alpha_8 = \alpha_I)$, $\left(\widetilde{\beta}_0 = \beta_0 = \beta_1\right)$, $\left(\widetilde{\beta}_1 = \beta_2 = \beta_3\right)$, $\left(\widetilde{\beta}_2 = \beta_4 = \beta_5\right)$, $\left(\widetilde{\beta}_3 = \beta_6 = \beta_7\right)$, $\left(\widetilde{\beta}_4 = \beta_8 = \beta_I\right)$.

- $M_6 : \boldsymbol{\theta}_{[6]} = (p, \phi, \alpha_0, \widetilde{\alpha}_1, \beta_0, \beta_1, ..., \beta_I)$ with $(\widetilde{\alpha}_1 = \alpha_1 = ... = \alpha_I)$.

Variable selection models

the joint posterior distribution $\pi(M_k, \boldsymbol{\theta}_{[k]} \mid \mathcal{D}_I)$, $\boldsymbol{\theta}_{[k]} = \left(\widetilde{\theta}_1, \widetilde{\theta}_2, ..., \widetilde{\theta}_{N_{[k]}}\right)$

a prior distribution $\pi(M_k)$ for the model

a prior for the parameters conditional on the model $\pi\left(\boldsymbol{\theta}_{[k]} \mid M_k\right)$

$$\pi(\boldsymbol{\theta}_{[m]} \mid M_k) = \prod_{i=1}^{N_{[m]}} \left[b_{\widetilde{\theta}_{i,[m]}} - a_{\widetilde{\theta}_{i,[m]}}\right]^{-1}$$

The modified version of Congdon's (2006), formula A.3

$$\pi(M_k \mid \mathcal{D}_I) = \int \pi(M_k, \boldsymbol{\theta}_{[k]} \mid \mathcal{D}_I) d\boldsymbol{\theta}_{[k]} = \int \pi(M_k \mid \boldsymbol{\theta}_{[k]}, \mathcal{D}_I) \pi(\boldsymbol{\theta}_{[k]} \mid \mathcal{D}_I) d\boldsymbol{\theta}_{[k]}$$

$$\approx \frac{1}{T - T_b} \sum_{j=T_b+1}^{T} \pi(M_k \mid \mathcal{D}_I, \boldsymbol{\theta}_{j,[k]})$$

$$= \frac{1}{T - T_b} \sum_{j=T_b+1}^{T} \frac{L_{\mathcal{D}_I}(M_k, \boldsymbol{\theta}_{j,[k]}) \prod_{k=0}^{K} \pi(\boldsymbol{\theta}_{j,[k]} \mid M_k) \pi(M_k)}{\sum_{m=0}^{K} L_{\mathcal{D}_I}(M_m, \boldsymbol{\theta}_{j,[m]}) \prod_{k=0}^{K} \pi(\boldsymbol{\theta}_{j,[k]} \mid M_m) \pi(M_m)}$$

$$= \frac{1}{T - T_b} \sum_{j=T_b+1}^{T} \frac{L_{\mathcal{D}_I}(M_k, \boldsymbol{\theta}_{j,[k]})}{\sum_{m=0}^{K} L_{\mathcal{D}_I}(M_m, \boldsymbol{\theta}_{j,[m]})}.$$

Variable selection models

	M_0	M_1	M_2	M_3	M_4	M_5	M_6
$\pi(M_k \mid D_I)$	0.71	4.19E-54	3.04E-43	1.03E-28	6.71E-20	2.17E-21	0.29
DIC	399	649	600	535	498	507	398
LHR $p - value$	1	2.76E-50	1.67E-40	3.53E-28	5.78E-21	3.03E-23	0.043

Table 4: Posterior model probabilities $\pi\left(M_k|D_I\right)$,
Deviance Information Criterion (DIC) and
Likelihood Ratio (LHR) p-values
for variable selection models $M_0, \ldots, M_6$

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Claims reserves

	Model Averaging	Model Selection for p
Estimated Reserves	$ER = \widehat{R}^{\mathrm{B}} = E[\widetilde{R}\vert\mathcal{D}_I]$	$ER_p = E[\widetilde{R}\vert\mathcal{D}_I, p]$
Process Variance	$PV = E\left[\sum \phi\,(\alpha_i\beta_j)^p \vert\, \mathcal{D}_I\right]$	$PV_p = E\left[\sum \phi\,(\alpha_i\beta_j)^p \vert\, \mathcal{D}_I, p\right]$
Estimation Error	$EE = \mathrm{Var}(\widetilde{R}\vert\mathcal{D}_I)$	$EE_p = \mathrm{Var}(\widetilde{R}\vert\mathcal{D}_I, p)$

Table 5: Quantities used for analysis of the claims reserving problem

$$ER = E[ER_p|\mathcal{D}_I],$$

$$PV = E[PV_p|\mathcal{D}_I],$$

$$EE = E[EE_p|\mathcal{D}_I] + \mathrm{Var}(ER_p|\mathcal{D}_I)$$

	Model Averaging	
Statistic	Bayesian Estimate	MLE Estimate
ER	624.1 (0.7)	602.630
$\sqrt{PV}$	37.3 (0.2)	25.937
$\sqrt{EE}$	44.8 (0.5)	28.336
$\sqrt{MSEP}$	58.3(0.5)	38.414

Table 6: Model averaged estimates

	Model Averaging	
VaR_q	R	$\widetilde{R}$
$\mathrm{VaR}_{75\%}$	659.8 (0.9)	650.6 (1.0)
$\mathrm{VaR}_{90\%}$	698.4 (1.2)	680.4 (1.3)
$\mathrm{VaR}_{95\%}$	724.0 (1.5)	701.7 (1.6)

Table 7: Bayesian model averaged estimates of Value at Risk

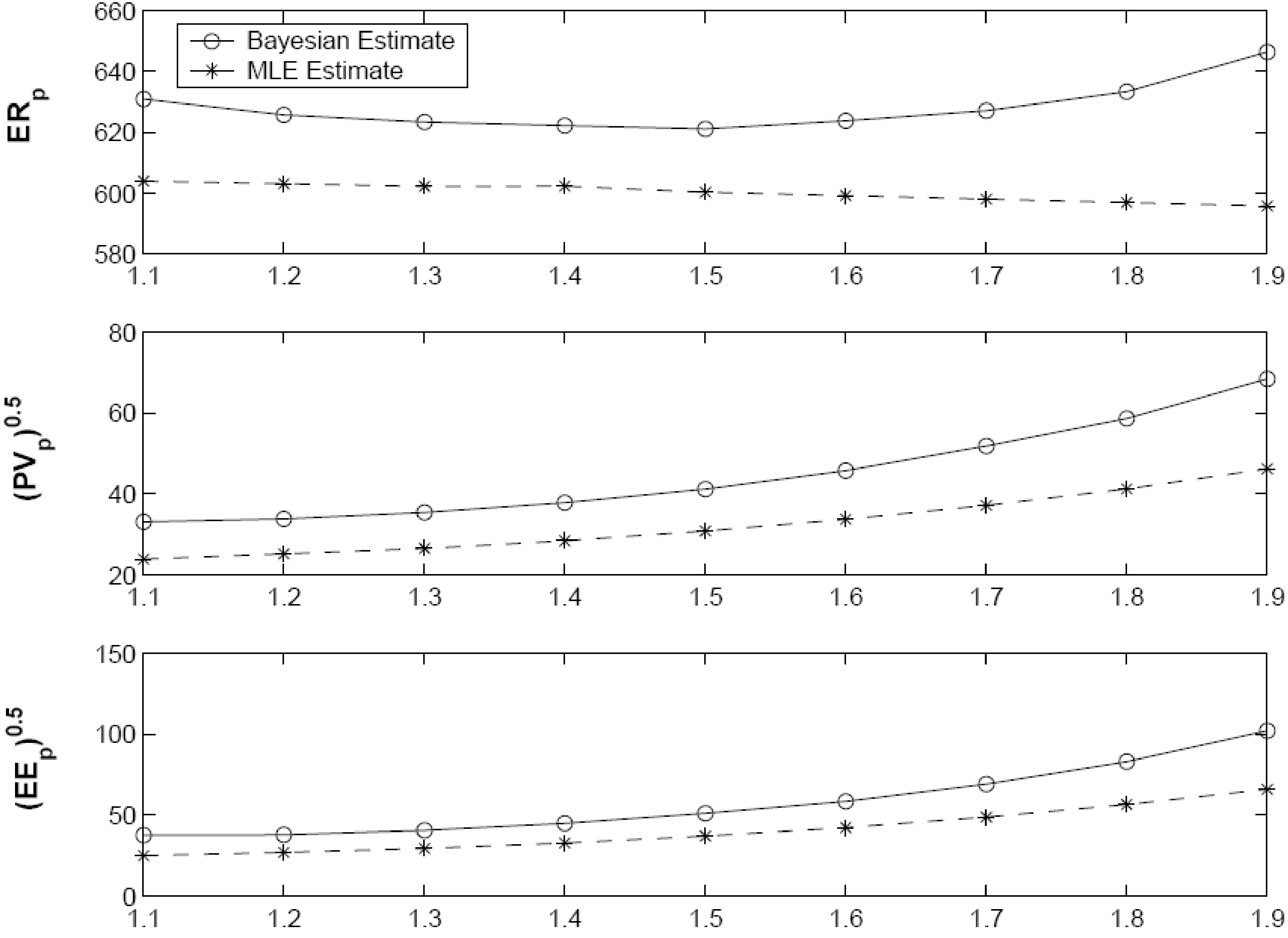

Figure 6: Estimates of quantities from Table 5 conditional on p.

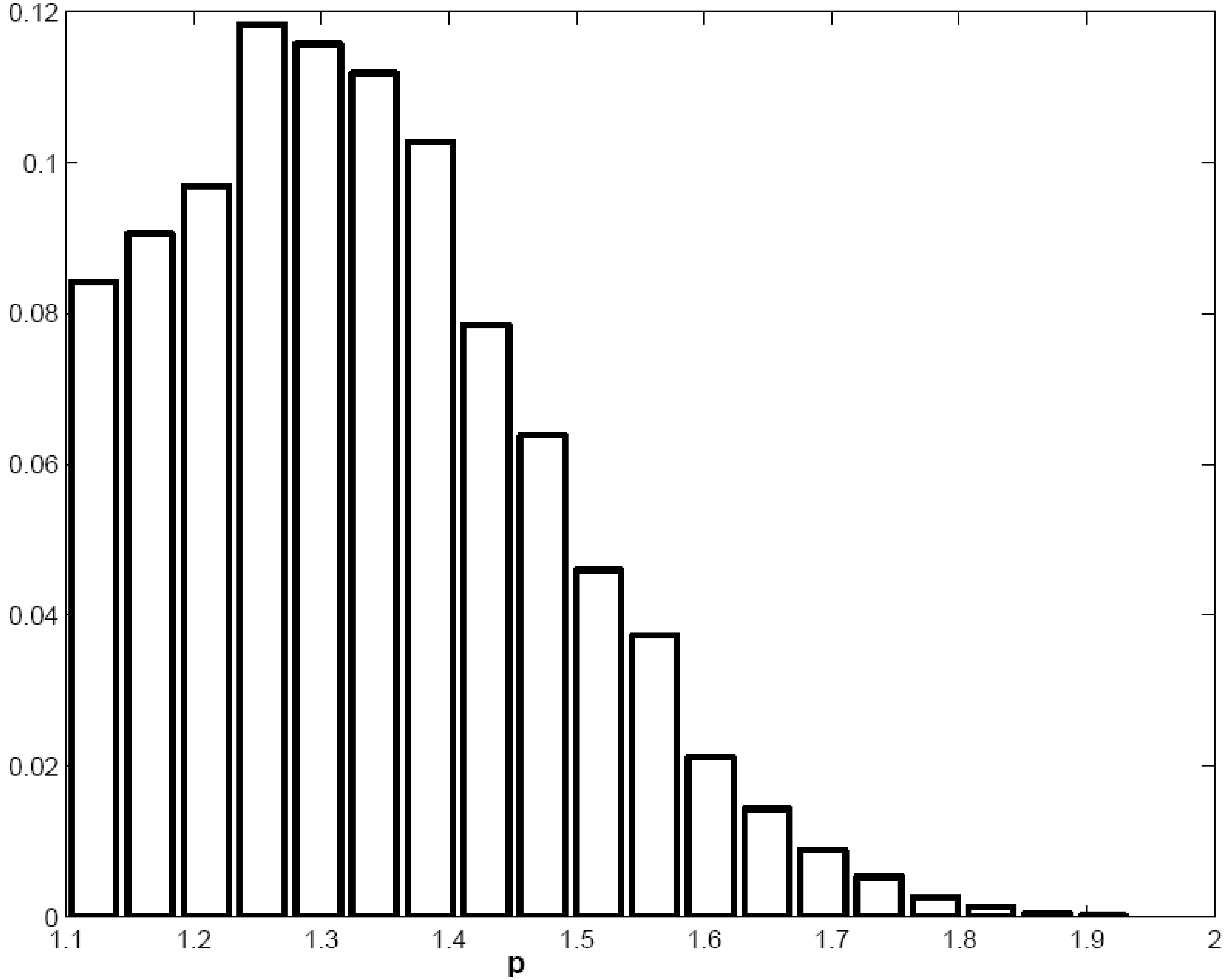

Figure 7: Posterior distribution of the model parameter p.

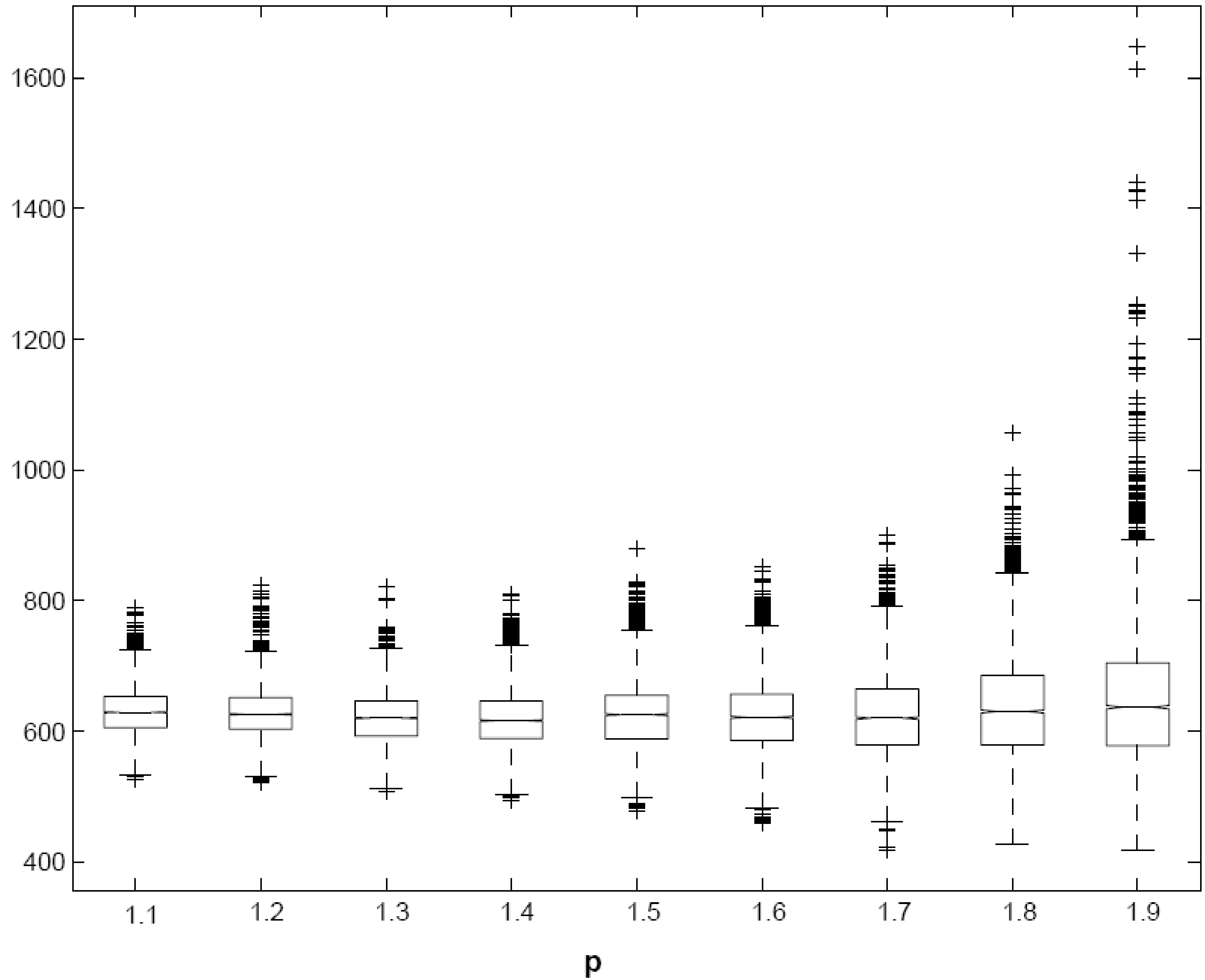

Figure 8: Predicted claim reserves $\widetilde{R}$ distributional summaries

Results: average over *p*

- Claims reserve MLE, $\widehat{R}^{\mathrm{MLE}}$, is less than Bayesian estimate $\widehat{R}^{\mathrm{B}}$ by approximately 3%, which is estimation bias of claims reserve MLE.
- $\sqrt{EE}$ and $\sqrt{PV}$ are of the same magnitude, approximately 6-7% of total reserve.
- MLEs for $\sqrt{EE}$ and $\sqrt{PV}$ are less than corresponding Bayesian estimates by approximately 37% and 30%, respectively.
- The difference between $\widehat{R}^{\mathrm{MLE}}$ and $\widehat{R}^{\mathrm{B}}$ is of the same order of magnitude as $\sqrt{EE}$ and $\sqrt{PV}$ and thus is significant.

Results: conditioning on *p*

- MLE of ER_p is almost constant, varying approximately from a maximum of 603.96 ($p = 1.1$) to minimum of 595.78 ($p = 1.9$) while the MLE for ER was 602.63.
- Bayesian estimates for ER_p changes as a function of p. Approximately, it ranged from a maximum of 646.4 ($p = 1.9$) to a minimum of 621.1 ($p = 1.5$) while the Bayesian estimator for ER was 624.1. Hence, the difference (estimation bias) within this possible model range is ≈ 25 which is of a similar order as process uncertainty and estimation error.

- Bayesian estimators for $\sqrt{PV_p}$ and $\sqrt{EE_p}$ increase as p increases approximately from 33.1 to 68.5 and from 37.4 to 102.0 respectively, while the Bayesian estimators for $\sqrt{PV}$ and $\sqrt{EE}$ are 37.3 and 44.8 correspondingly. Hence, the resulting risk measure strongly varies in p which has a large influence on quantitative solvency requirements. The MLEs for PV_p and EE_p are significantly less than the corresponding Bayesian estimators. Also, the difference between MLE and Bayesian estimators increases as p increases.

Conclusions

- Development of a Bayesian model for claims reserving under Tweedie's compound Poisson model – quantification process, estimation and model uncertainties; variable selection.
- MLEs are materially different from Bayesian estimators – posterior distributions are different from Gaussian.
- Future work: variable selection problem – Reversible Jump MCMC; considering model parameter p outside the (1, 2) range.

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Thank you