Pricing and hedging defaultable claims

Tomasz R. Bielecki, Illinois Institute of Technology, USA
Monique Jeanblanc, Université d'Évry Val d'Essonne, France
Marek Rutkowski, University of New South Wales, Australia

UT Vienna
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Some models

Intensity based models

The default time is a **stopping time** τ in some given filtration $\mathbf{G}$.

The intensity of τ is **a G-adapted** non-negative process λ such that

$$H_t - \int_0^{t\wedge\tau} \lambda_s ds = H_t - \int_0^t (1 - H_s)\lambda_s ds$$

is a $\mathbf{G}$-martingale, where $H_t = \mathbb{1}_{\tau \leq t}$.

Then, using integration by parts formula, for any $X \in \mathcal{G}_T$:

$$\mathbb{E}(X\,\mathbb{1}_{T<\tau}|\mathcal{G}_t) = \mathbb{1}_{\tau>t}\left(e^{\Lambda_t}\mathbb{E}(e^{-\Lambda_T}X|\mathcal{G}_t) - \mathbb{E}(e^{\Lambda_\tau}\,\Delta Y_\tau\,\mathbb{1}_{\tau\leq T}|\mathcal{G}_t)\right)$$

where $Y_t = \mathbb{E}\left(X\exp\left(-\Lambda_T\right)|\mathcal{G}_t\right)$ and $\Lambda_t = \int_0^t \lambda_u du$

The main difficulties are the following:

- In the definition of the intensity, the behavior of λ after time τ is not important.
- The jump size, depending on the choice of λ is difficult to compute.

From **Aven's lemma** $\lambda_t = \mathbb{P}(t < \tau \leq t + dt | \mathcal{G}_t)$.

We emphazise that, using this theorem gives a value of the intensity which is equal to 0 after the default time.

This is not convenient for using the previous pricing rule, since, with this choice of intensity, the process Y has a jump at time τ.

Example: Let us study the simple case where λ is a constant, and $X = 1$. Let us take, as the **G**-adapted intensity the process $\mathbb{1}_{t<\tau}\lambda$. Then,

$$\begin{aligned} Y_t &= \mathbb{E}\left(\exp -\lambda(T \wedge \tau)|\mathcal{G}_t\right) \\ &= \mathbb{1}_{t<\tau} e^{\lambda t} \int_t^\infty e^{-\lambda(u\wedge T)} \lambda e^{-\lambda u} du + \mathbb{1}_{\tau \leq t} e^{-\lambda\tau} \\ &= \mathbb{1}_{t<\tau} \left(e^{\lambda t} \int_t^T \lambda e^{-2\lambda u} du + e^{-\lambda T} \right) + \mathbb{1}_{\tau \leq t} e^{-\lambda\tau} \end{aligned}$$

The jump of Y at time τ is

$$\frac{1}{2} e^{-\lambda\tau} - e^{-\lambda T} + \frac{1}{2} e^{\lambda\tau} e^{-2\lambda T}$$

Cox process models

In this model

- A **filtration** $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$ and a r.v. Θ, independent of $\mathcal{F}_\infty$, with exponential law
- A non-negative **F-adapted process** $(\lambda_t, t \geq 0)$,

are given

The default time is defined as

$$\tau = \inf\{t : \Lambda_t \geq \Theta\}$$

where $\Lambda_t = \int_0^t \lambda_s ds$.

The conditional law is

$$\mathbb{P}(\tau \leq t|\mathcal{F}_t) = 1 - e^{-\Lambda_t}$$

Note that this process is increasing. The process

$$H_t - \Lambda_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} \lambda_s ds$$

is a **G**-martingale where the filtration **G** is defined as $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, the filtration **H** being generated by the process $H_t = \mathbb{1}_{\tau\leq t}$. For $X \in \mathcal{F}_T$:

$$\mathbb{E}(X\,\mathbb{1}_{T<\tau}|\mathcal{G}_t) = \mathbb{1}_{\tau>t}\, e^{\Lambda_t}\mathbb{E}(e^{-\Lambda_T} X|\mathcal{G}_t) = \mathbb{1}_{\tau>t}\, e^{\Lambda_t}\mathbb{E}(e^{-\Lambda_T} X|\mathcal{F}_t)\,.$$

Hazard process models

- A reference **filtration F**
- A **random time** τ

are given

- **H** is the filtration generated by the process $H_t = \mathbb{1}_{\tau \leq t}$
- $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$

Let $G_t = \mathbb{P}(\tau > t|\mathcal{F}_t)$ assumed to be **continuous** and satisfying $G_t > 0$.
Then, for $X \in \mathcal{F}_T$:

$$\mathbb{E}(X\mathbb{1}_{T<\tau}|\mathcal{G}_t) = \mathbb{1}_{\tau>t}G_t^{-1}\mathbb{E}(G_T X|\mathcal{F}_t)\,.$$

The super-martingale G admits a Doob-Meyer decomposition as

$$G_t = Z_t - A_t$$

where Z is an **F-martingale** and A a **predictable increasing** process. We assume that $A_t = \int_0^t a_s ds$ and we denote

$$\Lambda_t = \int_0^t \frac{a_s}{G_s} ds = \int_0^t \lambda_s ds$$

The process

$$M_t = H_t - \Lambda_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} \frac{a_s}{G_s} ds$$

is a $\mathbf{G}$-martingale. The process Λ is called the **F-intensity**.

The intensity rate $\lambda = aG^{-1}$ can be obtained as

$$\lambda_t = \lim_{h\to 0} \frac{1}{h} \frac{\mathbb{P}(t < \tau < t+h | \mathcal{F}_t)}{\mathbb{P}(t < \tau | \mathcal{F}_t)} .$$

In the particular case where $\mathbf{F}$ is the **trivial filtration**, $G_t = \mathbb{P}(\tau > t) = G(t)$ is an **decreasing function** (assumed to be differentiable) and

$$M_t = H_t - \Lambda_{t\wedge\tau} = H_t - \int_0^{t\wedge\tau} \frac{f(s)}{G(s)} ds$$

where $f(t) = -G'(t)$.

In some cases, **the intensity is not efficient to compute the value of the claim**:

- $e^{\Lambda_t}\mathbb{E}(Xe^{-\Lambda_T}|\mathcal{F}_t)$ can be different from $e^{\Gamma_t}\mathbb{E}(Xe^{-\Gamma_T}|\mathcal{F}_t)$
- $e^{\Lambda_t}\mathbb{E}(Xe^{-\Lambda_T}|\mathcal{G}_t)$ can be different from $e^{\Gamma_t}\mathbb{E}(Xe^{-\Gamma_T}|\mathcal{F}_t)$

Indeed, for $t = 0$, if the quantities $\mathbb{E}(Xe^{-\Lambda_T})$ and $\mathbb{E}(Xe^{-\Gamma_T})$ are equal for any $X \in \mathcal{F}_T$, this would imply $\Lambda_T = \Gamma_T$. This is the case if $Z = 0$ (which implies that $G = -A$ is decreasing)

If G is decreasing, then $Z = 0$.

If **F**-martingales are **G**-martingales (**H hypothesis**), then G is decreasing.

This is the case if the **F**-market is complete and arbitrage free, and if using **G**-adapted strategies in the **F**-market does not induce arbitrage opportunities.

We shall present a **simple and useful case** where the process G is not decreasing.

Two Correlated Default Times

We study the case of **two random times** τ_1, τ_2 with $\mathbb{P}(\tau_1 = \tau_2) = 0$.

We denote by

- $\tau = \inf(\tau_1, \tau_2)$,
- $(H_t^i, t \geq 0)$ the default process associated with $\tau_i, (i = 1, 2)$,
- $\mathbf{H}^i$ is the filtration generated by the process H^i: $\mathcal{H}_t^i = \sigma(\tau_i \wedge t)$
- $\mathbf{H}$ is the filtration generated by the processes H^i:

$$\mathcal{H}_t = \sigma(\tau_1 \wedge t) \vee \sigma(\tau_2 \wedge t) = \mathcal{H}_t^1 \vee \mathcal{H}_t^2 \,.$$

Computation of joint laws

We note G **the survival probability** of the pair (τ_1, τ_2), i.e.,

$$G(t,s) = \mathbb{P}(\tau_1 > t, \tau_2 > s)\,.$$

Then, $\mathbb{P}(t < \tau_1 | \tau_2) = h(t, \tau_2)$ where,

$$h(t,s) = \frac{\partial_2 G(t,s)}{\partial_2 G(0,s)}$$

Martingales

- **Filtration** $\mathbf{H}^i = \sigma(\tau_i \wedge t)$ The processes

$$M_t^i = H_t^i - \int_0^{t\wedge\tau_i} \frac{f_i(s)}{G_i(s)} ds$$

where

$$G_i(s) = \mathbb{P}(\tau_i > s) = \int_s^\infty f_i(u) du$$

are $\mathcal{H}_t^i$**-martingales**. In that case

$$G_1(t) = G(t, 0), \ f_1(t) = -\partial_1 G(t, 0)$$

- **Filtration $\mathbf{H} = \mathbf{H}^1 \vee \mathbf{H}^2$** The process

$$M_t^{(1)} = H_t^1 - \int_0^{t\wedge\tau_1} \frac{a_s^{(1)}}{G_s^{(1)}} ds$$

is a **H-martingale** where

- $a_t^{(1)} = -H_t^2 \partial_1 h^{(1)}(t, \tau_2) - (1 - H_t^2) \frac{\partial_1 G(t,t)}{G(0,t)}$
- $h^{(1)}(t, s) = \frac{\partial_2 G(t,s)}{\partial_2 G(0,s)}$
- $G_t^{(1)} = \mathbb{P}(\tau_1 > t | \mathcal{H}_t^2)$ is a $\mathbf{H}^2$-submartingale, it is NOT decreasing

PROOF: Some easy computation enables us to write

$$
\begin{aligned}
1 - G_t^{(1)} = F_t^{(1)} &= H_t^2 \mathbb{P}(\tau_1 \leq t | \tau_2) + (1 - H_t^2) \frac{\mathbb{P}(\tau_1 \leq t < \tau_2)}{\mathbb{P}(\tau_2 > t)} \\
&= H_t^2 (1 - h^{(1)}(t, \tau_2)) + (1 - H_t^2) \frac{G(0,t) - G(t,t)}{G(0,t)}
\end{aligned}
$$

where

$$
h^{(1)}(t,v) = \frac{\partial_2 G(t,v)}{\partial_2 G(0,v)} \, .
$$

It follows that

$$
\begin{aligned}
dF_t^{(1)} &= \left(\frac{G(t,t)}{G(0,t)} - \frac{\partial_2 G(t,t)}{\partial_2 G(0,t)} \right) dM_t^2 \\
&\quad - \left(H_t^2 \partial_1 h^{(1)}(t, \tau_2) + (1 - H_t^2) \frac{\partial_1 G(t,t)}{G(0,t)} \right) dt
\end{aligned}
$$

The hazard process has a non null martingale part. This shows again that the intensity is not the good tool to work with.

Hedging Defaultable Claims

Defaultable claim

A **defaultable claim maturing at** T is a quadruple (X, A, Z, τ), where

- X is an $\mathcal{F}_T$-measurable random variable,
- A is an $\mathbf{F}$-adapted process of finite variation
- Z is an $\mathbf{F}$-predictable process.

The **dividend process** D of a defaultable claim maturing at T equals

$$D_t = X \mathbb{1}_{\{\tau > T\}} \mathbb{1}_{[T,\infty[}(t) + \int_{]0,t\wedge T]} (1 - H_u)\, dA_u + \int_{]0,t\wedge T]} Z_u\, dH_u.$$

First approach: PDE

Two default free assets, one total default asset

We assume that 3 (non paying dividends) assets are traded, with price's dynamics

$$\begin{aligned} dY_t &= Y_t r dt \\ dY_t^2 &= Y_t^2(\mu_{2,t}dt + \sigma_{2,t}dW_t) \\ dY_t^3 &= Y_{t-}^3(\mu_{3,t}dt + \sigma_{3,t}dW_t - dM_t)\,. \end{aligned}$$

Note that $Y_t^3 = 0$ on the set $t \geq \tau$.

There exists a unique e.m.m. as soon as $\sigma_1 \neq \sigma_2$ and the risk-neutral intensity γ^* satisfies

$$\gamma^* = \gamma + \mu_3 - r + (r - \mu_2)\frac{\sigma_2}{\sigma_3} > 0$$

PDE Approach, payment at maturity

We are working in a model with constant (or Markovian) coefficients

$$\begin{aligned} dY_t &= Y_t r dt \\ dY_t^2 &= Y_t^2(\mu_2 dt + \sigma_2 dW_t) \\ dY_t^3 &= Y_{t-}^3(\mu_3 dt + \sigma_3 dW_t - dM_t)\,. \end{aligned}$$

Let $C(t, Y_t^2, Y_t^3, H_t)$ be the price of the contingent claim $G(Y_T^2, Y_T^3, H_T)$ and γ^* be the risk-neutral intensity of default.

The process $e^{-rt}C(t, Y_t^2, Y_t^3, H_t)$ is a martingale under the e.m.m.

Then,

$$\partial_t C(t,y_2,y_3;0) + r y_2 \partial_2 C(t,y_2,y_3;0) + r^* y_3 \partial_3 C(t,y_2,y_3;0) - r^* C(t,y_2,y_3;0)$$
$$+ \frac{1}{2}\sum_{i,j=2}^{3} \sigma_i \sigma_j y_i y_j \partial_{ij} C(t,y_2,y_3;0) + \gamma^* C(t,y_2,0;1) = 0$$

where $r^* = r + \gamma^*$ and

$$\partial_t C(t,y_2;1) + r y_2 \partial_2 C(t,y_2;1) + \frac{1}{2}\sigma_2^2 y_2^2 \partial_{22} C(t,y_2;1) - r C(t,y_2;1) = 0$$

with the terminal conditions

$$C(T,y_2,y_3;0) = G(y_2,y_3;0), \quad C(T,y_2;1) = G(y_2,0;1).$$

The ***replicating strategy*** ϕ *for* Y *is given by formulae*

$$\begin{aligned}
\phi_t^3 Y_{t-}^3 &= -C(t, Y_t^2, 0; 1) + C(t, Y_t^2, Y_{t-}^3; 0) = -\Delta C(t) \\
\sigma_2 \phi_t^2 Y_t^2 &= -\Delta C(t) + \sum_{i=2}^{3} Y_{t-}^i \sigma_i \partial_i C(t) \\
\phi_t^1 Y_t^1 &= C(t) - \phi_t^2 Y_t^2 - \phi_t^3 Y_t^3 \, .
\end{aligned}$$

Note that, in the case of survival claim, $C(t, Y_t^2, 0; 1) = 0$ and $\phi_t^3 Y_{t-}^3 = C(t, Y_{t-}^2, Y_{t-}^3; 0)$ for every $t \in [0, T]$. Hence, the following relationships holds, for every $t < \tau$,

$$\phi_t^3 Y_t^3 = C(t, Y_t^2, Y_t^3; 0), \qquad \boxed{\phi_t^1 Y_t^1 + \phi_t^2 Y_t^2 = 0}$$

The last equality is a special case of the **balance condition**. It ensures that the wealth of a replicating portfolio falls to 0 at default time.

Example Let us consider a survival claim $Y = \mathbb{1}_{\{\tau > T\}} G(Y_T^2)$. Its pre-default pricing function $C(\cdot, 0)$ does not depend on y_3, and satisfies the PDE

$$\partial_t C(t,y,0) + ry\partial_2 C(t,y,0) + \frac{1}{2}\sigma^2 y^2 \partial_{22} C(t,y,0) - r^* C(t,y,0) = 0$$

with the terminal condition $C(T,y,0) = G(y)$. The present set-up covers the case of a vulnerable option written on a default-free asset Y^2. The solution is $e^{(r^*-r)t}\tilde{C}(t,y) = e^{\gamma^* t}\tilde{C}(t,y)$ where $\widetilde{C}$ is solution of BS equation with interest rate r:

$$\begin{aligned} r\widetilde{C}(t,y,0) &= \partial_t \widetilde{C}(t,y,0) + ry\partial_y \widetilde{C}(t,y,0) + \frac{1}{2}\sigma^2 y^2 \partial_{yy}\widetilde{C}(t,y,0) \\ \widetilde{C}(T,y) &= e^{-\gamma^* T} G(y) \end{aligned}$$

Dividend-paying Assets

A probability space $(\Omega, \mathbf{G}, \mathbb{P})$ is given. All the processes are assumed to be $\mathbf{G}$-adapted and càdlàg.

We denote by

$$B_t = \exp(\int_0^t r(s)ds)$$

the savings account, where r is deterministic.

Self-Financing Trading Strategies

We assume that S^i, $i = 1, \ldots, k$ are non-dividend-paying assets.
Let $S = S^0$ denote the price process of a security that pays dividends according to a process of finite variation D, with $D_0 = 0$.
The **wealth process** associated to the strategy $\phi = (\phi^0, \phi^1, \ldots, \phi^k)$ is

$$V_t(\phi) = \sum_{i=0}^{k} \phi_t^i S_t^i \,.$$

A strategy ϕ is said to be **self-financing** if

$$V_t(\phi) = V_0(\phi) + \int_{]0,t]} \phi_u^0 \, dD_u + \sum_{i=0}^{k} \int_{]0,t]} \phi_u^i \, dS_u^i.$$

We say that $\mathbb{Q}$, equivalent to $\mathbb{P}$, is a **martingale measure** if

- the discounted price $B_t^{-1}S_t^i$ of any non-dividend paying traded security is a $\mathbb{Q}$-martingale
- the process $S_t B_t^{-1} + \int_{]0,t]} B_u^{-1}\, dD_u$ is a $\mathbb{Q}$-martingale

Here, the ex-dividend price process S associated with the dividend process D satisfies:

$$S_t = B_t\, \mathbb{E}_Q\Big(S_T B_T^{-1} + \int_{]t,T]} B_u^{-1}\, dD_u \,\Big|\, \mathcal{G}_t\Big).$$

For any self-financing trading strategy ϕ, the discounted wealth process $B_t^{-1}V_t(\phi)$ is a $\mathbb{Q}$-martingale.

Credit Default Swap

A **credit default swap** with a constant rate κ and *recovery at default* is a defaultable claim $(0, A, Z, \tau)$, where

- $Z_t \equiv \delta(t)$
- $A_t = -\kappa\, t$ for every $t \in [0, T]$.

The function (or process) $\delta : [0, T] \to I\!R$ represents the **default protection**, and the constant $\kappa \in I\!R$ represents the CDS rate (also termed the **spread, premium** or *annuity* of a CDS).

Deterministic Intensity

We assume here that **F is the trivial filtration**. Let

$$G(t) = \mathbb{Q}(\tau > t) = \int_t^\infty f(u)du$$

be the $\mathbb{Q}$-survival probability. In that case, for any function h,

$$\mathbb{E}(h(\tau)|\mathcal{H}_t)\mathbb{1}_{\{t<\tau\}} = -\mathbb{1}_{\{t<\tau\}}\frac{1}{G(t)}\mathbb{E}\left(\int_t^\infty h(u)dG(u)\right)$$

We assume that $r = 0$.

The ex-dividend price of a CDS maturing at T with rate κ is given by the formula

$$\begin{aligned} S_t(\kappa) &= \mathbb{E}_Q\Big(\mathbb{1}_{\{t<\tau\leq T\}}\delta(\tau)\,\Big|\,\mathcal{H}_t\Big) - \mathbb{E}_Q\Big(\mathbb{1}_{\{t<\tau\}}\kappa\big((\tau\wedge T)-t\big)\,\Big|\,\mathcal{H}_t\Big) \\ &= \mathbb{1}_{\{t<\tau\}}\,\frac{1}{G(t)}\left(-\int_t^T \delta(u)\,dG(u) - \kappa\int_t^T G(u)\,du\right). \end{aligned}$$

Note that this price can take negative values.

Let us define the $(\mathbb{Q}, \mathbf{H})$-martingale

$$M_t = H_t - \int_0^t (1 - H_u)\gamma(u)\, du,$$

where $\gamma(u) = -\frac{G'(u)}{G(u)}$.

The dynamics of the ex-dividend price $S_t(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1 - H_t)(\kappa - \delta(t)\gamma(t))\, dt,$$

The function $\widetilde{S}_t(\kappa)$ such that $\mathbb{1}_{\{t<\tau\}}\widetilde{S}_t(\kappa) = \mathbb{1}_{\{t<\tau\}}S_t(\kappa)$ is the **predefault-price**, it satisfies

$$d\widetilde{S}_t(\kappa) = \Big(\widetilde{S}_t(\kappa)\gamma(t) + (\kappa - \delta(t)\gamma(t))\Big)\, dt,$$

We assume that $\widetilde{S}_t(\kappa) \neq \delta(t)$ for every $t \in [0, T]$.

Hedging with CDS

Let

$$Y = \mathbb{1}_{\{T \geq \tau\}} h(\tau) + \mathbb{1}_{\{T < \tau\}} c(T)$$

be a defaultable claim. Then the self-financing strategy $\phi = (\phi^0, \phi^1)$ is **a replicating strategy for** Y where

$$\phi^0(t) = \frac{h(t) - \widetilde{g}(t)}{\delta(t) - \widetilde{S}_t(\kappa)},$$

$$\widetilde{g}(t) = \frac{1}{G(t)} \left(\int_0^t h(s) dG(s) + c(T) G(T) \right)$$

and $\phi_t^1 = V_t(\phi) - \phi_t^0 S_t(\kappa)$, where $V_t(\phi) = \mathbb{E}_Q(Y|\mathcal{H}_t)$.

Proof: The terminal value of the wealth is

$$V_T = h(\tau)\mathbb{1}_{\tau<T} + c(T)\mathbb{1}_{T<\tau}$$

On the one hand

$$\begin{aligned} E(V_T|\mathcal{H}_t) = V_t &= h(\tau)\mathbb{1}_{\tau\leq t} + \mathbb{1}_{\tau<t}\frac{1}{G(t)}\left(XG(T) + \int_0^t h(s)dG(s)\right) \\ &= \int_0^t h(s)dH_s + (1-H_s)\frac{1}{G(t)}\left(c(T)\,G(T) + \int_0^t h(s)dG(s)\right) \end{aligned}$$

hence $dV_t = (h(t) - \widetilde{g}(t))\,dM_t$ with

$$\widetilde{g}(t) = \frac{1}{G(t)}(\int_0^t h(s)dG(s) + c(T)G(T))$$

On the other hand, $dV_t = \phi_t^0(S_{t-}(\kappa) - \delta(t))\,dM_t$.

Stochastic intensity

We now assume that some **reference filtration** $\mathbf{F}$ is given. We set $\mathbf{G} = \mathbf{F} \vee \mathbf{H}$ so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for every $t \in \mathbb{R}_+$. The filtration $\mathbf{G}$ is referred to as to the **full filtration**.

We define the **survival process**

$$G_t = \mathbb{Q}\{\tau > t \,|\, \mathcal{F}_t\}\,.$$

We assume that G is a continuous process with $G_0 = 1$ and $G_t > 0$.

We recall the key formulae

- For $Y \in \mathcal{F}_T$

$$\mathbb{E}_Q(\mathbb{1}_{\{T<\tau\}} Y \,|\, \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_Q(G_T Y \,|\, \mathcal{F}_t)\,.$$

- for Z predictable

$$\mathbb{E}_Q(Z_\tau \mathbb{1}_{t<\tau\leq T} | \mathcal{G}_t) = \mathbb{1}_{\{t<\tau\}} \frac{1}{G_t} \mathbb{E}_Q(-\int_t^T Z_u dG_u \,|\, \mathcal{F}_t)\,.$$

The ex-dividend price of a credit default swap, with a rate κ and a protection payment δ_τ at default, equals, for every $t \in [s, T]$,

$$\begin{aligned} S_t(\kappa) &= \mathbb{E}_Q\Big(\mathbb{1}_{\{t<\tau\leq T\}}\delta_\tau \,\Big|\, \mathcal{G}_t\Big) - \mathbb{E}_Q\Big(\mathbb{1}_{\{t<\tau\}}\kappa((\tau\wedge T)-t)\Big) \,\Big|\, \mathcal{G}_t\Big), \\ &= \mathbb{1}_{\{t<\tau\}}\,\frac{1}{G_t}\,\mathbb{E}_Q\left(-\int_t^T \delta_u\, dG_u + \kappa\int_t^\infty ((u\wedge T)-t)dG_u \,\Big|\, \mathcal{F}_t\right). \end{aligned}$$

We now assume that (H) hypothesis holds between F and G.

(H) hypothesis states that **F**-martingales are **G**-martingales.

It is known that if the **F** market is complete and arbitrage free, and if using **G**-adapted strategies in the **F**-market does not induce arbitrage opportunities, then this hypothesis holds.

Under (H) hypothesis, the process G is decreasing. We assume that G is absolutely continuous wrt Lebesgue measure. The process

$$M_t = H_t - \int_0^{t\wedge\tau} \gamma_u \, du,$$

with $\gamma_t dt = -\frac{dG_t}{G_t}$ is a **G**-martingale.

The dynamics of the CDS ex-dividend price $S_t(\kappa)$ are

$$dS_t(\kappa) = -S_{t-}(\kappa)\, dM_t + (1-H_t) B_t G_t^{-1}\, dm_t + (1-H_t)(r_t S_t(\kappa) + \kappa - \delta_t \gamma_t)\, dt,$$

where m is the $(\mathbb{Q}, \mathbf{F})$-martingale given by

$$m_t = \mathbb{E}_Q \left(\int_0^T B_u^{-1} \delta_u G_u \gamma_u \, du - \kappa \int_0^T B_u^{-1} G_u \, du \,\middle|\, \mathcal{F}_t \right).$$

using two CDS with maturities T_i, rates κ_i and protection payment δ^i.
We assume $r = 0$.

Our aim is to hedge

$$Y = \mathbb{1}_{\{T \geq \tau\}} Z_\tau + \mathbb{1}_{\{T < \tau\}} X.$$

The ex-dividend price of Y is

$$Y_t \quad = \quad \mathbb{1}_{\{t<\tau\}} \, {G_t}^{-1} \, \mathbb{E}_{Q^*} \left(G_T X + \int_t^T G_u Z_u \lambda_u \, du \, \Big| \, \mathcal{F}_t \right)$$

and its dynamics are

$$dY_t = -Y_{t-} \, dM_t - (1 - H_t) \lambda_t Z_t \, dt + (1 - H_t) G_t^{-1} \, dm_t^Z$$

with $m_t^Z = \mathbb{E}_Q(-\int_0^\infty Z_u dG_u + G_T X | \mathcal{F}_t)$

Let ζ_t^i defined as

$$m_t^i = \mathbb{E}_Q\left(\int_0^T \delta_u^i G_u \gamma_u \, du - \kappa_i \int_0^T G_u \, du \,\Big|\, \mathcal{F}_t\right), \; dm_t^i = \zeta_t^i dW_t$$

and

$$m_t^Z = \mathbb{E}_Q(-\int_0^\infty Z_u dG_u + G_T X | \mathcal{F}_t), \; dm_t^Z = \zeta_t^Z dW_t$$

Assume that there exist $\mathbf{F}$-predictable processes ϕ^0, ϕ^1 such that

$$\sum_{i=0}^{1} \phi_t^i \big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big) = Z_t - \widetilde{g}_t, \quad \sum_{i=0}^{1} \phi_t^i \zeta_t^i = \zeta_t,$$

where $\widetilde{g}$ is given by

$$\widetilde{g}_t = \frac{1}{G_t}\, \mathbb{E}_Q \left(- \int_t^T Z_u \, dG_u + G_T X \,\Big|\, \mathcal{F}_t \right).$$

Let $\phi_t^2 = V_t(\phi) - \sum_{i=0}^{1} \phi_t^i S_t^i(\kappa_i)$, where the process $V(\phi)$ is given by

$$dV_t(\phi) = \sum_{i=0}^{1} \phi_t^i \big(dS_t^i(\kappa_i) + dD_t^i \big)$$

with the initial condition $V_0(\phi) = \mathbb{E}_Q(Y)$. Then the self-financing trading strategy $\phi = (\phi^0, \phi^1, \phi^2)$ is admissible and it is a replicating strategy for a defaultable claim $(X, 0, Z, \tau)$.

Defaults Correlation: First case

We assume here that F is the trivial filtration. We now assume that two CDS's with default times τ_1 and τ_2 are given. The function G is the survival probability of the pair (τ_1, τ_2). We assume that the pair (τ_1, τ_2) admits a density f. Some easy computation lead to $\mathbb{Q}(t < \tau_1 | \tau_2) = h(t, \tau_2)$ where:

$$h(t,s) = \frac{\partial_2 G(t,s)}{\partial_2 G(0,s)}$$

It is rather easy to find the dynamics of a CDS S^1 associated with default τ_1, with parameters (δ, κ). One starts from the fact that, on the set $\{\tau_1 > t, \tau_2 > t\}$

$$\begin{aligned} S_t^1 &= \frac{1}{G(t,t)} \left(- \int_t^T \delta(u) G(du, t) - \kappa \int_t^T du\, G(u,t) \right) \\ &= V^1(t) \end{aligned}$$

and, on the set $\{\tau_1 > t > \tau_2\}$

$$\begin{aligned} S_t^1 &= \frac{1}{\partial_2 G(t, \tau_2)} \left(- \int_t^T du\, \delta(u) f(u, \tau_2) - \kappa \int_t^T du\, \partial_2 G(u, \tau_2) \right) \\ &= V^2(t, \tau_2) \end{aligned}$$

Hence the dynamics of the pre-default price $\widetilde{S}^1_t(\kappa_1)$ are

$$d\widetilde{S}^1_t(\kappa_1) = (\widetilde{\gamma}_1(t) + \widetilde{\gamma}_2(t))\widetilde{S}^1_t(\kappa_1)\, dt + \left(\kappa_1 - \delta_1(t)\widetilde{\gamma}_1(t) - S^1_{t|2}(\kappa_1)\widetilde{\gamma}_2(t)\right) dt,$$

where

$$S^1_{t|2}(\kappa_1) = \frac{\int_t^T \delta_1(u) f(u,t) du}{\int_t^\infty f(u,t)\, du} - \kappa_1 \frac{\int_t^T du \int_u^\infty dz f(z,t)}{\int_t^\infty f(u,t)\, du}.$$

and, on the set $\tau_2 < t \leq \tau_1$

$$dS^1_t(\kappa_1) = \left(\lambda^{1|2}(t,\tau_2)\left(S^1_t(\kappa_1) - \delta_1(t)\right) + \kappa_1\right) dt$$

where $\lambda^{1|2}(t,s) = -\frac{f(t,s)}{\partial_2 G(t,s)}$, evaluated at $s = \tau_2$, represents the value of the default intensity process of τ_1 with respect to $\mathbf{H}^2$ on the event $\{\tau_2 < t\}$.

Hedging of Credit Spreads and Default Correlations

In this section, we assume that $\mathbf{F}$ is a Brownian filtration and that the interest rate is null. Our aim is to obtain the dynamics of a CDS in the simple case where two different credit names are considered.

Joint Survival Process

Hence we assume that we are given two strictly positive random times τ_1 and τ_2. We introduce the **conditional joint survival process** $G(u,v;t)$

$$G(u,v;t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t).$$

We write

$$\partial_1 G(u,v;t) = \frac{\partial}{\partial u} G(u,v;t), \quad \partial_{12} G(u,v;t) = \frac{\partial^2}{\partial u \partial v} G(u,v;t).$$

We assume that the density $f(u,v;t) = \partial_{12}G(u,v;t)$ with respect to u and v exists, so that

$$G(u,v;t) = \int_u^\infty \Big(\int_v^\infty f(x,y;t)\,dy \Big)\,dx.$$

For any fixed $(u,v) \in I\!R_+^2$, the $\mathbf{F}$-martingale
$G(u,v;t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v \,|\, \mathcal{F}_t)$ admits the integral representation

$$G(u,v;t) = \mathbb{Q}(\tau_1 > u, \tau_2 > v) + \int_0^t g(u,v;s)\,dW_s$$

where $g(u,v;s)$ is some $\mathbf{F}$-predictable process.

Let us introduce the filtrations $\mathbf{H}^i, \mathbf{H}, \mathbf{G}^i$ and $\mathbf{G}$ associated with default times by setting

$$\mathcal{H}_t^i = \sigma(H_s^i;\, s \in [0,t]), \quad \mathcal{H}_t = \mathcal{H}_t^1 \vee \mathcal{H}_t^2, \quad \mathcal{G}_t^i = \mathcal{F}_t \vee \mathcal{H}_t^i, \quad \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t,$$

where $H_t^i = \mathbb{1}_{\{\tau_i \leq t\}}$. We assume that the usual conditions of completeness and right-continuity are satisfied by these filtrations.

We assume that immersion hypothesis holds between $\mathbf{F}$ and $\mathbf{G}$. In particular, any $\mathbf{F}$-martingale is also a $\mathbf{G}^i$-martingale for $i = 1, 2$.

In general, there is no reason that any $\mathbf{G}^i$-martingale is a $\mathbf{G}$-martingale. (see a specific case in Ehlers and Schonbucher)

Valuation of Single-Name CDSs

Let us now examine the valuation of single-name CDSs under the assumption that the interest rate equals zero.

We consider the CDS

- with the constant spread κ_1,
- which delivers $\delta_1(\tau_1)$ at time τ_1 if $\tau_1 < T_1$, where δ_1 is a deterministic function.

The value $S^1(\kappa_1)$ of this CDS, computed in the filtration $\mathbf{G}$, i.e., taking care on the information on the second default contained in that filtration, is computed in two successive steps.

On the set $t < \tau_{(1)} = \tau_1 \wedge \tau_2$, the ex-dividend price of the CDS equals, on the event $\{t < \widetilde{1}\}$,

$$S_t^1(\kappa_1) = \widetilde{S}_t^1(\kappa_1) = \frac{1}{G(t,t;t)} \left(- \int_t^{T_1} \delta_1(u) \partial_1 G(u,t;t)\, du - \kappa_1 \int_t^{T_1} G(u,t;t)\, du \right).$$

On the event $\{\tau_2 \leq t < \tau_1\}$, we have that

$$S_t^1(\kappa_1) = \frac{1}{\partial_2 G(t,\tau_2;t)} \left(- \int_t^{T_1} \delta_1(u) f(u,\tau_2;t)\, du - \kappa_1 \int_t^{T_1} \partial_2 G(u,\tau_2;t)\, du \right).$$

Price Dynamics of Single-Name CDSs

Let us return to the study of a general case. By applying the Itô-Wentzell theorem, we get

$$\begin{aligned} G(u,t;t) &= G(u,0;0) + \int_0^t g(u,s;s)\,dW_s + \int_0^t \partial_2 G(u,s;s)\,ds \\ G(t,t;t) &= G(0,0;0) + \int_0^t g(s,s;s)\,dW_s + \int_0^t (\partial_1 G(s,s;s) + \partial_2 G(s,s;s))\,ds. \end{aligned}$$

The cumulative price $S^{c,1}(\kappa_1)$ satisfies, on $[0, T \wedge \tau_{(1)}]$,

$$dS_t^{c,1}(\kappa_1) = (\delta_1(t) - \widetilde{S}_t^1(\kappa_1))\, d\widehat{M}_t^1 + (S_{t|2}^1(\kappa_1) - \widetilde{S}_t^1(\kappa_1))\, d\widehat{M}_t^2$$
$$- \frac{1}{G(t,t;t)} \Big(\int_t^{T_1} \delta_1(u) \partial_1 g(u,t;t)\, du + \kappa_1 \int_t^{T_1} g(u,t;t)\, du \Big) dW_t\,.$$

Here

$$\widehat{M}_t^i = H_{t\wedge\tau_{(1)}}^i - \int_0^{t\wedge\tau_{(1)}} \widetilde{\lambda}_u^i\, du,$$

is a $\mathbf{G}$-martingale, where $\widetilde{\lambda}_t^i = -\frac{\partial_i G(t,t;t)}{G(t,t;t)}$ is the **pre-default intensity** for the ith name

Replication of a First-to-Default Claim

A **first-to-default claim** with maturity T is a defaultable claim $(X, A, Z, \tau_{(1)})$ where

- X is an $\mathcal{F}_T$-measurable amount payable at maturity if no default occurs
- $A : [0, T] \to \mathbb{R}$ with $A_0 = 0$ represents the dividend stream up to $\tau_{(1)}$,
- $Z = (Z^1, Z^2, \ldots, Z^n)$ is the vector of **F**-predictable, real-valued processes, where $Z^i_{\tau_{(1)}}$ specifies the recovery received at time $\tau_{(1)}$ if the ith name is the first defaulted name, that is, on the event $\{\tau_i = \tau_{(1)} \leq T\}$.

The cumulative price S^{cum} is given by

$$dS_t^{cum} = r_t S_t^{cum}\,dt + \sum_{i=1}^{n}(Z_t^i - S_{t-})\,d\widehat{M}_t^i + (1-H_t^{(1)})B_t(G_{(1)}(t;t))^{-1}\,dm_t,$$

where the $\mathbf{F}$-martingale m is given by the formula

$$\begin{aligned} m_t &= \mathbb{E}_{Q^*}\left(B_T^{-1}G_{(1)}(T;T)X + \sum_{i=1}^{n}\int_0^T B_u^{-1}G_{(1)}(u;u)Z_u^i\widetilde{\lambda}_u^i\,du \right. \\ &\quad \left. - \int_0^T B_u^{-1}G_{(1)}(u;u)\,dA_u \,\Big|\, \mathcal{F}_t \right). \end{aligned}$$

The pre-default ex-dividend price satisfies

$$d\widetilde{S}_t = (r_t + \widetilde{\lambda}_t)\widetilde{S}_t\,dt - \sum_{i=1}^{n} \widetilde{\lambda}_t^i Z_t^i\,dt + dA_t + B_t(G(t,t;t))^{-1}\,dm_t.$$

Since $\mathbf{F}$ is generated by a Brownian motion, there exists an $\mathbf{F}$-predictable process ζ such that

$$d\widetilde{S}_t = (r_t + \widetilde{\lambda}_t)\widetilde{S}_t\,dt - \sum_{i=1}^{n} \widetilde{\lambda}_t^i Z_t^i\,dt + dA_t + B_t(G(t,t;t))^{-1}\zeta_t\,dW_t.$$

We say that a self-financing strategy $\phi = (\phi^0, \phi^1, \ldots, \phi^k)$ **replicates** a first-to-default claim $(X, A, Z, \tau_{(1)})$ if its wealth process $V(\phi)$ satisfies the equality $V_{t\wedge\tau_{(1)}}(\phi) = S_{t\wedge\tau_{(1)}}$ for any $t \in [0, T]$.

We have, for any $t \in [0, T]$,

$$dV_t(\phi) = r_t V_t(\phi)\, dt + \sum_{i=1}^{n} \phi_t^i \Big(\big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big)\, dM_t^i + \sum_{j=1\,,j\neq i}^{n} \big(S_{t|j}^i - \widetilde{S}_t^i(\kappa_i)\big)\, dM_t^j + (1 - H_t) B_t (G(t,t;t))^{-1}\, dn_t^i \Big)$$

where

$$n_t^i = \mathbb{E}_{Q^*}\left(\int_0^{T_i} \frac{G(u,u;u)}{B_u} \Big(\delta_u^i \widetilde{\lambda}_u^i + \sum_{j=1, j\neq i}^{n} S_{u|j}^i \widetilde{\lambda}_u^j \Big)\, du - \kappa_i \int_0^{T_i} \frac{G(u,u;u)}{B_u}\, du \,\Big|\, \mathcal{F}_t \right).$$

Let $\widetilde{\phi}_t = (\widetilde{\phi}^1_t, \widetilde{\phi}^2_t, \ldots, \widetilde{\phi}^n_t)$ be a solution to the following equations

$$\widetilde{\phi}^i_t\big(\delta^i_t - \widetilde{S}^i_t(\kappa_i)\big) + \sum_{j=1,\, j\neq i}^{n} \widetilde{\phi}^j_t\big(S^j_{t|i}(\kappa_j) - \widetilde{S}^j_t(\kappa_j)\big) = Z^i_t - \widetilde{S}_t$$

and $\sum_{i=1}^n \widetilde{\phi}^i_t \zeta^i_t = \zeta_t$.

Let us set $\phi^i_t = \widetilde{\phi}^i(\tau_{(1)} \wedge t)$ for $i = 1, 2, \ldots, n$ and $t \in [0, T]$.

Then the self-financing trading strategy $\phi = (B^{-1}(V(\phi) - \phi \cdot S), \ldots, \phi^k)$ replicates the first-to-default claim $(X, A, Z, \tau_{(1)})$.

Replication with Market CDSs

When considering trading strategies involving CDSs issued in the past, one encounters a practical difficulty regarding their liquidity. For this reason, we shall now analyze trading strategies based on market CDS contracts. Recall that for each maturity T_i by the CDS issued at time t we mean the CDS over $[t, T]$ with the spread $\kappa(t, T_i)$. We shall briefly write κ_t^i rather than $\kappa(t, T_i)$. We will now define a contract – that we call a **market CDS** — which at any time t has similar features as the T_i-maturity CDS issued at this date t, in particular, it has the ex-dividend price equal to zero.

A T_i**-maturity market CDS** has the dividend process equal to ${}^*D^i = {}^*D(\kappa_i, \delta^i, T_i, \tau)$ where, for $t \in [0, T_i]$,

$$ {}^*D^i_t = \int_{]0,t]} B_u \, d(B_u^{-1} S^i_u(\kappa_i)) + D^i_t,$$

where $D^i = D(\kappa_i, \delta^i, T_i, \tau)$ for some fixed spread κ_i.

The ex-dividend price ${}^*S^i$ **of the** T_i**-maturity market CDS equals zero for any** $t \in [0, T_i]$.

Since market CDSs are traded on the ex-dividend basis, to describe the self-financing trading strategies in the savings account B and the market CDSs with ex-dividend prices ${}^*S^i$.

A strategy $\phi = (\phi^0, \ldots, \phi^k)$ in the savings account B and the market CDSs with dividends ${}^*D^i$ is said to be *self-financing* if its wealth $V_t(\phi) = \phi_t^0 B_t$ satisfies $V_t(\phi) = V_0(\phi) + G_t(\phi)$ for every $t \in [0, T]$, where the gains process $G(\phi)$ is defined as follows

$$G_t(\phi) = \int_{]0,t]} \phi_u^0 \, dB_u + \sum_{i=1}^{k} \int_{]0,t]} \phi_u^i \, d\,{}^*D_u^i.$$

Let ϕ be a self-financing strategy in the savings account B and ex-dividend prices $S^i(\kappa_i)$, $i = 1, \ldots, k$. Then the strategy $\psi = (\psi^0, \ldots, \psi^k)$ where $\psi^i = \phi^i$ for $i = 1, \ldots, k$ and $\psi^0_t = B_t^{-1} V_t(\phi)$ is a self-financing strategy in the savings account B and the market CDSs with dividends ${}^*D^i$ and its wealth process satisfies $V(\psi) = V(\phi)$.

The cumulative price of the T_i-maturity market CDS satisfies

$$\begin{aligned} {}^*S^{c,i}_t &= {}^*S^i_t + B_t \int_{]0,t]} B_u^{-1} \, d\, {}^*D^i_u \\ &= \mathbb{1}_{\{t<\tau\}} (\kappa^i_t - \kappa_i) \widetilde{A}(t,T) - B_t S^i_0(\kappa_i) + B_t \int_{]0,t]} B_u^{-1} \, dD^i_u \end{aligned}$$

where

$$\widetilde{A}(t,T) = \frac{B_t}{G_t} \, \mathbb{E}_{Q^*} \left(\int_t^{T \wedge \tau} B_u^{-1} du \, \Big| \, \mathcal{F}_t \right).$$

If we choose $\kappa_i = \kappa_0^i$ then

$$^*S_t^{c,i} = \mathbb{1}_{\{t<\tau\}}(\kappa_t^i - \kappa_0^i)\widetilde{A}(t,T) + B_t \int_{]0,t]} B_u^{-1}\, dD_u^i = S_t^{c,i}(\kappa_0^i).$$

Assume that there exist $\mathbf{F}$-predictable processes $\phi^1, \ldots, \phi^k$ satisfying the following conditions, for any $t \in [0, T]$,

$$\sum_{i=1}^{k} \phi_t^i \big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big) = Z_t - \widetilde{S}_t, \quad \sum_{i=1}^{k} \phi_t^i \zeta_t^i = \xi_t.$$

Let the process $V(\phi)$ be given by

$$dV_t(\phi) = \sum_{i=1}^{k} \phi_t^i \Big(\big(\delta_t^i - \widetilde{S}_t^i(\kappa_i)\big)\, dM_t + (1 - H_t) B_t G_t^{-1} dn_t^i \Big)$$

with the initial condition $V_0(\phi) = Y_0$ and let ϕ^0 be given by, for $t \in [0, T]$,

$$\phi_t^0 = B_t^{-1} V_t(\phi).$$

Then the self-financing trading strategy $\phi = (\phi^0, \ldots, \phi^k)$ in the savings account B and market CDSs with dividends ${}^*D^i$, $i = 1, \ldots, k$ replicates the defaultable claim (X, A, Z, τ).

Intuitively, one can think of the market CDS as a stream of CDSs that are continuously entered into and immediately unwound. Consequently, one can assume an accounting convention according to which one never holds a non-market CDS: suppose at time 0 one goes long the market CDS with spread κ_0^i. If one still owns it at time $t > 0$, the convention dictates that one owns at time t the market CDS with spread κ_t^i, but that it has already paid the cumulative dividends. In this way, we avoid any problem with considering the short-sale positions: what would be a short-sale position in an on-the-run (i.e., non-market) CDS becomes a short position in the corresponding market CDS. This mathematical convention is actually consistent with the market practice where default protection is bought or sold and then nullified, that is, CDSs are longed or shorted and then unwound, as needed.

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Thank you for your attention