Fenchel conjugation for set–valued convex functions and applications to set–valued risk measures

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Let X, V be locally convex spaces and $K \subseteq V$ a closed convex pointed cone generating a partial order in V. Denote by $\widehat{\mathcal{P}}(V)$ the power set of V (including the empty set). We prove that every proper closed and convex function $f : X \to \widehat{\mathcal{P}}(V)$ has a set–valued affine minorant of a certain type and every such function is the pointwise supremum of its (set–valued) affine minorants. Moreover, a biconjugation theorem is valid for suitable defined Fenchel conjugates.

However, there is a series of serious questions: What do properness, closedness and convexity mean for set–valued functions? What is a set–valued affine minorant, in what sense does it minorize a set–valued function and how shall we define the pointwise supremum of such objects?

Trying to answer these questions we shall make use of order relations in $\widehat{\mathcal{P}}(V)$. For $M_1, M_2 \subseteq V$, by

$$M_1 \preccurlyeq_K M_2 \quad \Longleftrightarrow \quad M_2 \subseteq M_1 + K$$

a quasiorder is introduced that replaces the usual $\leq$–relation in the real–valued case. The set–valued substitutes for linear functionals have the form

$$T_{(x^*,v^*)}(x) := \{v \in V : \ x^*(x) + v^*(v) = 0\}$$

with $x^* \in X^*$ and $v^* \in K^*$, the (negative) dual cone of K. A Fenchel conjugate $f^* : X^* \times K^* \to \widehat{\mathcal{P}}(V)$ is defined by

$$-f^*(x^*; v^*) := \bigcup_{x \in X} \left\{ f(x) + T_{(-x^*,v^*)}(x) + K \right\}$$

where the union can be interpreted as an infimum w.r.t. $\preccurlyeq_K$.

Finally, applications will be discussed to set–valued convex and coherent measures of risk which are a recent development in financial mathematics in order to obtain a representation theorem via biconjugation as in the real–valued case.